CSE 421: Introduction to Algorithms

Graph Traversal

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Undirected Graph  \( G = (V,E) \)
Directed Graph $G = (V,E)$
Graph Traversal

- Learn the basic structure of a graph
- Walk from a fixed starting vertex $s$ to find all vertices reachable from $s$
Generic Graph Traversal Algorithm

Find: set \( R \) of vertices reachable from \( s \in V \)

Reachable(\( s \)):
\[
R \leftarrow \{ s \} \\
\text{While there is a } (u,v) \in E \text{ where } u \in R \text{ and } v \notin R \\
\quad \text{Add } v \text{ to } R \\
\text{Return } R
\]
Claim: At termination $R$ is the set of nodes reachable from $s$

Proof

$\subseteq$: For every node $v \in R$ there is a path from $s$ to $v$

$\supseteq$: Suppose there is a node $w \notin R$ reachable from $s$ via a path $P$

- Take first node $v$ on $P$ such that $v \notin R$
- Predecessor $u$ of $v$ in $P$ satisfies
  - $u \in R$
  - $(u,v) \in E$

But this contradicts the fact that the algorithm exited the while loop.
Graph Traversal

- Learn the basic structure of a graph
- Walk from a fixed starting vertex $s$ to find all vertices reachable from $s$

Three states of vertices
- unvisited
- visited/discovered (in $R$)
- fully-explored (in $R$ and all neighbors in $R$)
Breadth-First Search

- Completely explore the vertices in order of their distance from \textbf{s}

- Naturally implemented using a queue
BFS(s)

Global initialization: mark all vertices “unvisited”
BFS(s)
mark s “visited”; \( R \leftarrow \{s\} \); layer \( L_0 \leftarrow \{s\} \)
while \( L_i \) not empty
\( L_{i+1} \leftarrow \emptyset \)
For each \( u \in L_i \)
for each edge \( \{u,v\} \)
if (\( v \) is “unvisited”)
mark \( v \) “visited”
Add \( v \) to set \( R \) and to layer \( L_{i+1} \)
mark \( u \) “fully-explored”
i \leftarrow i+1
Properties of BFS($v$)

- **BFS($s$)** visits $x$ if and only if there is a path in $G$ from $s$ to $x$.

- Edges followed to undiscovered vertices define a “breadth first spanning tree” of $G$.

- Layer $i$ in this tree, $L_i$
  - those vertices $u$ such that the shortest path in $G$ from the root $s$ is of length $i$.

- On undirected graphs
  - All non-tree edges join vertices on the same or adjacent layers.
Properties of BFS

- On undirected graphs
  - All non-tree edges join vertices on the same or adjacent layers

- Suppose not
  - Then there would be vertices \((x,y)\) such that \(x \in L_i\) and \(y \in L_j\) and \(j > i + 1\)
  - Then, when vertices incident to \(x\) are considered in BFS \(y\) would be added to \(L_{i+1}\)
    and not to \(L_j\)
BFS Application: Shortest Paths

Tree gives shortest paths from start vertex

can label by distances from start vertex
Graph Search Application: Connected Components

- Want to answer questions of the form:
  - *Given*: vertices $u$ and $v$ in $G$
  - Is there a path from $u$ to $v$?
Graph Search Application: Connected Components

- Want to answer questions of the form:
  - **Given**: vertices $u$ and $v$ in $G$
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- **Idea**: create array $A$ such that
  
  $A[u] =$ smallest numbered vertex that is connected to $u$

Graph Search Application: Connected Components

- Want to answer questions of the form:
  - Given: vertices $u$ and $v$ in $G$
  - Is there a path from $u$ to $v$?

- Idea: create array $A$ such that $A[u] = \text{smallest numbered vertex that is connected to } u$

Q: Why not create an array $Path[u,v]$?
Graph Search Application: Connected Components

- initial state: all $v$ unvisited
  for $s \leftarrow 1$ to $n$ do
    if state($s$) $\neq$ “fully-explored” then
      BFS($s$): setting $A[u] \leftarrow s$ for each $u$ found
      (and marking $u$ visited/fully-explored)
    endif
  endfor

- Total cost: $O(n+m)$
  - each vertex is touched once in this outer
    procedure and the edges examined in the different
    BFS runs are disjoint
  - works also with Depth First Search
DFS(u) – Recursive version

Global Initialization: mark all vertices "unvisited"

DFS(u)

mark u “visited” and add u to R
for each edge \{u,v\}
    if (v is “unvisited”)
        DFS(v)
end for
mark u “fully-explored”
Properties of DFS(s)

- Like BFS(s):
  - DFS(s) visits x if and only if there is a path in G from s to x
  - Edges into undiscovered vertices define a "depth first spanning tree" of G

- Unlike the BFS tree:
  - the DFS spanning tree isn't minimum depth
  - its levels don't reflect min distance from the root
  - non-tree edges never join vertices on the same or adjacent levels

- BUT…
Non-tree edges

- All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

- No cross edges.
No cross edges in DFS on undirected graphs

- **Claim:** During **DFS(x)** every vertex marked visited is a descendant of **x** in the DFS tree **T**

- **Claim:** For every **x,y** in the DFS tree **T**, if **(x,y)** is an edge not in **T** then one of **x** or **y** is an ancestor of the other in **T**

- **Proof:**
  - One of **x** or **y** is visited first, suppose WLOG that **x** is visited first and therefore **DFS(x)** was called before **DFS(y)**
    - During **DFS(x)**, the edge **(x,y)** is examined
  - Since **(x,y)** is a not an edge of **T**, **y** was visited when the edge **(x,y)** was examined during **DFS(x)**
  - Therefore **y** was visited during the call to **DFS(x)** so **y** is a descendant of **x**.
Applications of Graph Traversal: Bipartiteness Testing

- **Easy**: A graph $G$ is not bipartite if it contains an odd length cycle
- **WLOG**: $G$ is connected
  - Otherwise run on each component
- **Simple idea**: start coloring nodes starting at a given node $s$
  - Color $s$ red
  - Color all neighbors of $s$ blue
  - Color all their neighbors red
  - If you ever hit a node that was already colored
    - the same color as you want to color it, ignore it
    - the opposite color, output error
BFS gives Bipartiteness

- Run BFS assigning all vertices from layer $L_i$ the color $i \mod 2$
  - i.e. red if they are in an even layer, blue if in an odd layer

- If there is an edge joining two vertices from the same layer then output “Not Bipartite”
Why does it work?

\[ s \]

\[ L_i \]

\[ u \] and \[ v \] have a common ancestor

Cycle length \( 2(j-i)+1 \)
DFS(v) for a directed graph
DFS(v)
Properties of Directed DFS

- Before $\text{DFS}(s)$ returns, it visits all previously unvisited vertices reachable via directed paths from $s$

- Every cycle contains a back edge in the DFS tree
Directed Acyclic Graphs

- A directed graph $G=(V,E)$ is **acyclic** if it has no directed cycles.

- **Terminology**: A directed acyclic graph is also called a **DAG**.
Topological Sort

- **Given:** a directed acyclic graph (DAG) $G=(V,E)$
- **Output:** numbering of the vertices of $G$ with distinct numbers from 1 to $n$ so edges only go from lower number to higher numbered vertices

**Applications**
- nodes represent tasks
- edges represent precedence between tasks
- topological sort gives a sequential schedule for solving them
Directed Acyclic Graph
In-degree 0 vertices

- Every DAG has a vertex of in-degree 0
- **Proof:** By contradiction
  - Suppose every vertex has some incoming edge
  - Consider following procedure:
    ```
    while (true) do
        v←some predecessor of v
    ```
  - After **n+1** steps where \( n=|V| \) there will be a repeated vertex
    - This yields a cycle, contradicting that it is a DAG
Topological Sort

- Can do using DFS

- Alternative simpler idea:
  - Any vertex of in-degree 0 can be given number 1 to start
  - Remove it from the graph and then give a vertex of in-degree 0 number 2, etc.
Topological Sort
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Implementing Topological Sort

- Go through all edges, computing array with in-degree for each vertex \( O(m+n) \)
- Maintain a queue (or stack) of vertices of in-degree 0
- Remove any vertex in queue and number it
- When a vertex is removed, decrease in-degree of each of its neighbors by 1 and add them to the queue if their degree drops to 0

Total cost \( O(m+n) \)