CSE 421: Introduction to Algorithms

Graph Traversal

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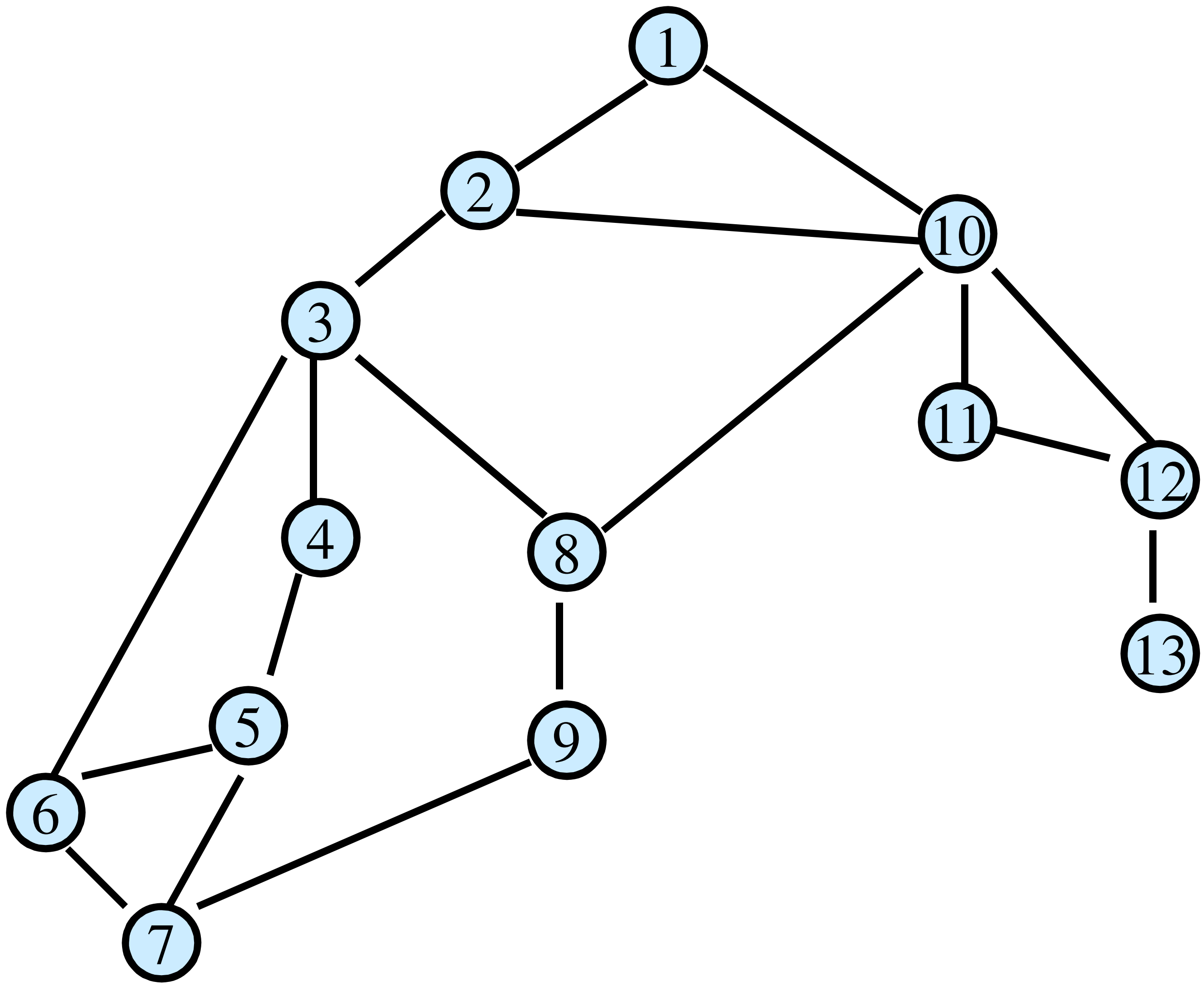
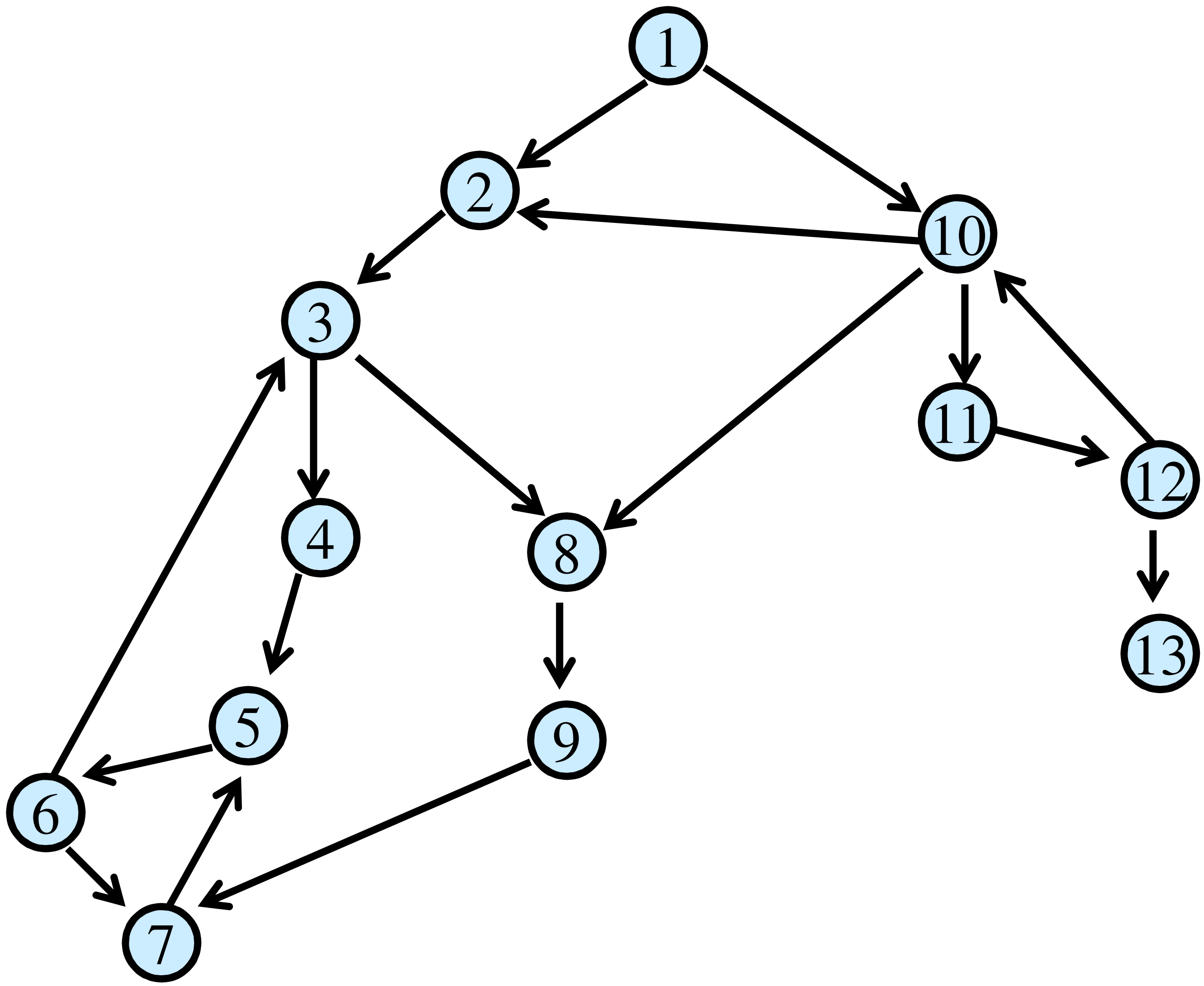
Undirected Graph   G = (V,E)



Directed Graph G = (V,E)



Graph Traversal

- Learn the basic structure of a graph
- Walk from a fixed starting vertex **s** to find all vertices reachable from **s**

Generic Graph Traversal Algorithm

Find: set **R** of vertices reachable from $\mathbf{s} \in \mathbf{V}$

Reachable(**s**):

$\mathbf{R} \leftarrow \{\mathbf{s}\}$

While there is a $(\mathbf{u},\mathbf{v}) \in \mathbf{E}$ where $\mathbf{u} \in \mathbf{R}$ and $\mathbf{v} \notin \mathbf{R}$

Add **v** to **R**

Return **R**

Generic Traversal Always Works

- Claim: At termination $\mathbf{R}$ is the set of nodes reachable from $\mathbf{s}$
- Proof
  - $\subseteq$: For every node $\mathbf{v} \in \mathbf{R}$ there is a path from $\mathbf{s}$ to $\mathbf{v}$
  - $\supseteq$: Suppose there is a node $\mathbf{w} \notin \mathbf{R}$ reachable from $\mathbf{s}$ via a path $\mathbf{P}$
    - Take first node $\mathbf{v}$ on $\mathbf{P}$ such that $\mathbf{v} \notin \mathbf{R}$
    - Predecessor $\mathbf{u}$ of $\mathbf{v}$ in $\mathbf{P}$ satisfies
      - $\mathbf{u} \in \mathbf{R}$
      - $(\mathbf{u},\mathbf{v}) \in \mathbf{E}$
    - But this contradicts the fact that the algorithm exited the while loop.

Graph Traversal

- Learn the basic structure of a graph
- Walk from a fixed starting vertex **s** to find all vertices reachable from **s**

- Three states of vertices
  - **unvisited**
  - **visited/discovered** (in **R**)
  - **fully-explored** (in **R** and all neighbors in **R**)

Breadth-First Search

- Completely explore the vertices in order of their distance from **s**

- Naturally implemented using a queue

BFS(s)

Global initialization: mark all vertices "unvisited"

BFS(**s**)

    mark **s** "visited"; **R**$\leftarrow${**s**}; layer $\mathbf{L_0}\leftarrow${**s**}

    while $\mathbf{L_i}$ not empty

        $\mathbf{L_{i+1}} \leftarrow \varnothing$

        For each $\mathbf{u}\in \mathbf{L_i}$

            for each edge {**u**,**v**}

                if (**v** is "unvisited")

                    mark **v** "visited"

                    Add **v** to set **R** and to layer $\mathbf{L_{i+1}}$

            mark **u** "fully-explored"

        $\mathbf{i} \leftarrow \mathbf{i+1}$

Properties of BFS(v)

- **BFS(s)** visits **x** if and only if there is a path in **G** from **s** to **x**.
- Edges followed to undiscovered vertices define a "breadth first spanning tree" of **G**
- Layer **i** in this tree, L_i
  - those vertices **u** such that the shortest path in **G** from the root **s** is of length **i**.
- On undirected graphs
  - All non-tree edges join vertices on the same or adjacent layers

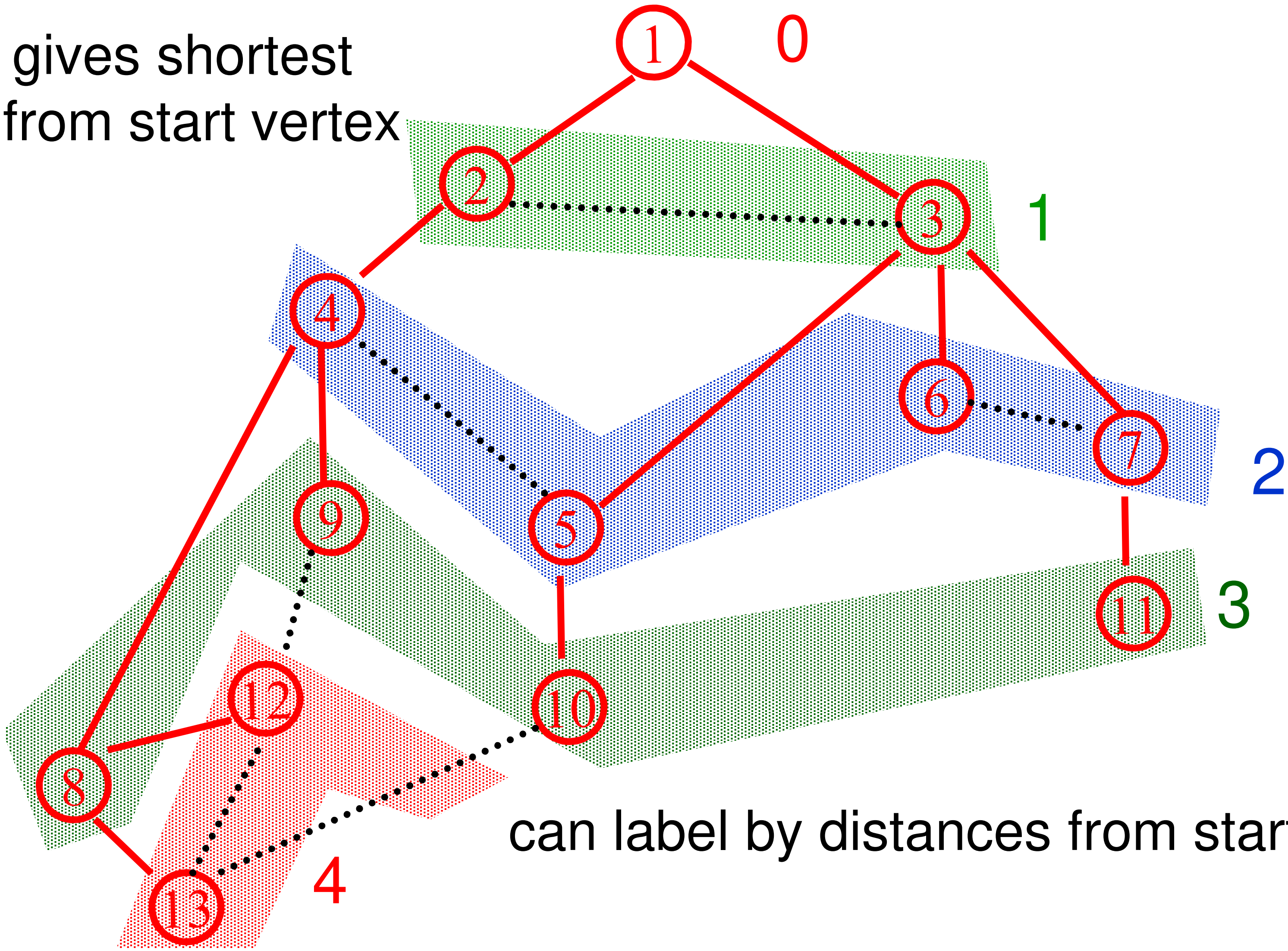
Properties of BFS

- On undirected graphs
  - All non-tree edges join vertices on the same or adjacent layers

  - Suppose not
    - Then there would be vertices $(\mathbf{x},\mathbf{y})$ such that $\mathbf{x} \in \mathbf{L}_i$ and $\mathbf{y} \in \mathbf{L}_j$ and $\mathbf{j} > \mathbf{i+1}$
    - Then, when vertices incident to $\mathbf{x}$ are considered in BFS $\mathbf{y}$ would be added to $\mathbf{L}_{i+1}$ and not to $\mathbf{L}_j$

BFS Application: Shortest Paths

Tree gives shortest paths from start vertex



can label by distances from start

Graph Search Application: Connected Components

- Want to answer questions of the form:
  - Given: vertices **u** and **v** in **G**
  - Is there a path from **u** to **v**?

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- Idea: create array **A** such that

  A[u] = smallest numbered vertex that is connected to **u**

  - question reduces to whether **A[u]=A[v]**?

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  - question reduces to whether **A**[**u**]=**A**[**v**]?

Q: Why not create an array **Path**[**u**,**v**]?

Graph Search Application: Connected Components

- initial state: all **v** unvisited
  for **s**$\leftarrow$**1** to **n** do
      if state(**s**) $\neq$ "fully-explored" then
          BFS(**s**): setting **A**[**u**] $\leftarrow$**s** for each **u** found
              (and marking **u** visited/fully-explored)
      endif
  endfor

- Total cost: **O(n+m)**
  - each vertex is touched once in this outer procedure and the edges examined in the different BFS runs are disjoint
  - works also with Depth First Search

DFS(u) – Recursive version

Global Initialization: mark all vertices "unvisited"

DFS(**u**)

mark **u** “visited” and add **u** to **R**

for each edge {**u**,**v**}

if (**v** is “unvisited”)

DFS(**v**)

end for

mark **u** “fully-explored”

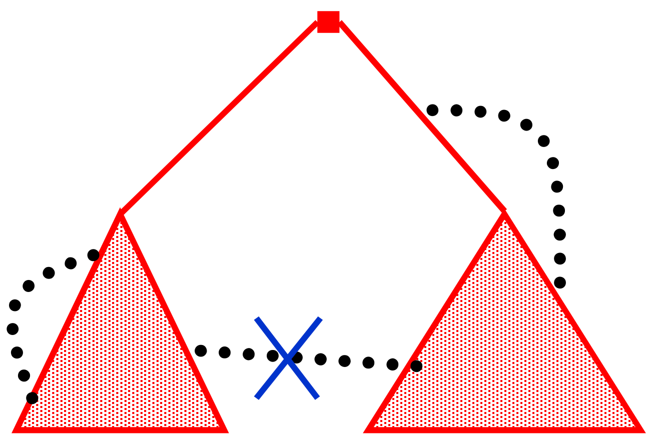
Properties of DFS(s)

- Like BFS(**s**):
  - DFS(**s**) visits **x** if and only if there is a path in **G** from **s** to **x**
  - Edges into undiscovered vertices define a "depth first spanning tree" of **G**
- Unlike the BFS tree:
  - the DFS spanning tree isn't minimum depth
  - its levels don't reflect min distance from the root
  - non-tree edges never join vertices on the same or adjacent levels
- BUT…

Non-tree edges

- All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

- No cross edges.



No cross edges in DFS on undirected graphs

- Claim: During **DFS(x)** every vertex marked visited is a descendant of **x** in the DFS tree **T**
- Claim: For every **x,y** in the DFS tree **T**, if **(x,y)** is an edge not in **T** then one of **x** or **y** is an ancestor of the other in **T**
- Proof:
  - One of **x** or **y** is visited first, suppose WLOG that **x** is visited first and therefore **DFS(x)** was called before **DFS(y)**
    - During **DFS(x)**, the edge **(x,y)** is examined
  - Since **(x,y)** is a not an edge of **T**, **y** was visited when the edge **(x,y)** was examined during **DFS(x)**
  - Therefore **y** was visited during the call to **DFS(x)** so **y** is a descendant of **x**.

Applications of Graph Traversal: Bipartiteness Testing

- Easy: A graph G is not bipartite if it contains an odd length cycle
- WLOG: G is connected
  - Otherwise run on each component
- Simple idea: start coloring nodes starting at a given node s
  - Color s red
  - Color all neighbors of s blue
  - Color all their neighbors red
  - If you ever hit a node that was already colored
    - the same color as you want to color it, ignore it
    - the opposite color, output error

BFS gives Bipartiteness

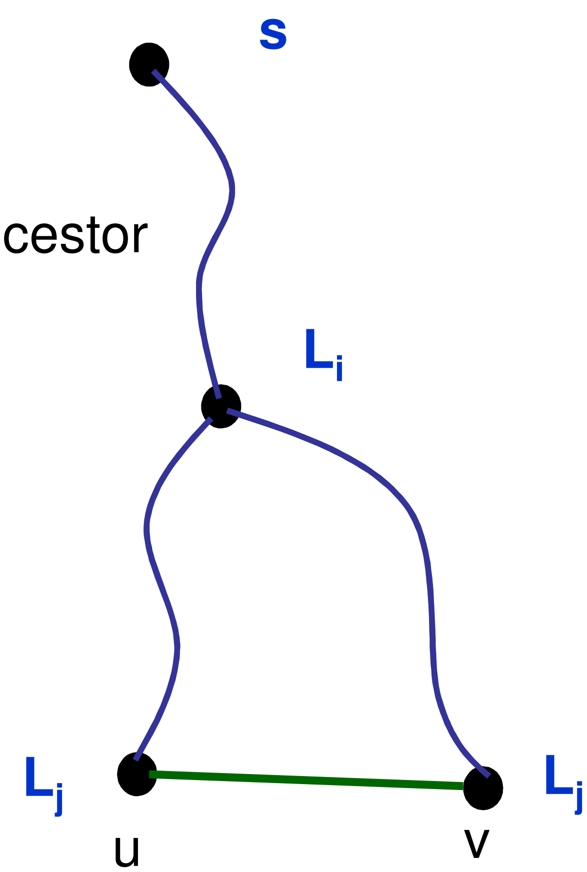
- Run BFS assigning all vertices from layer L_i the color $i \bmod 2$
  - i.e. red if they are in an even layer, blue if in an odd layer

- If there is an edge joining two vertices from the same layer then output “Not Bipartite”

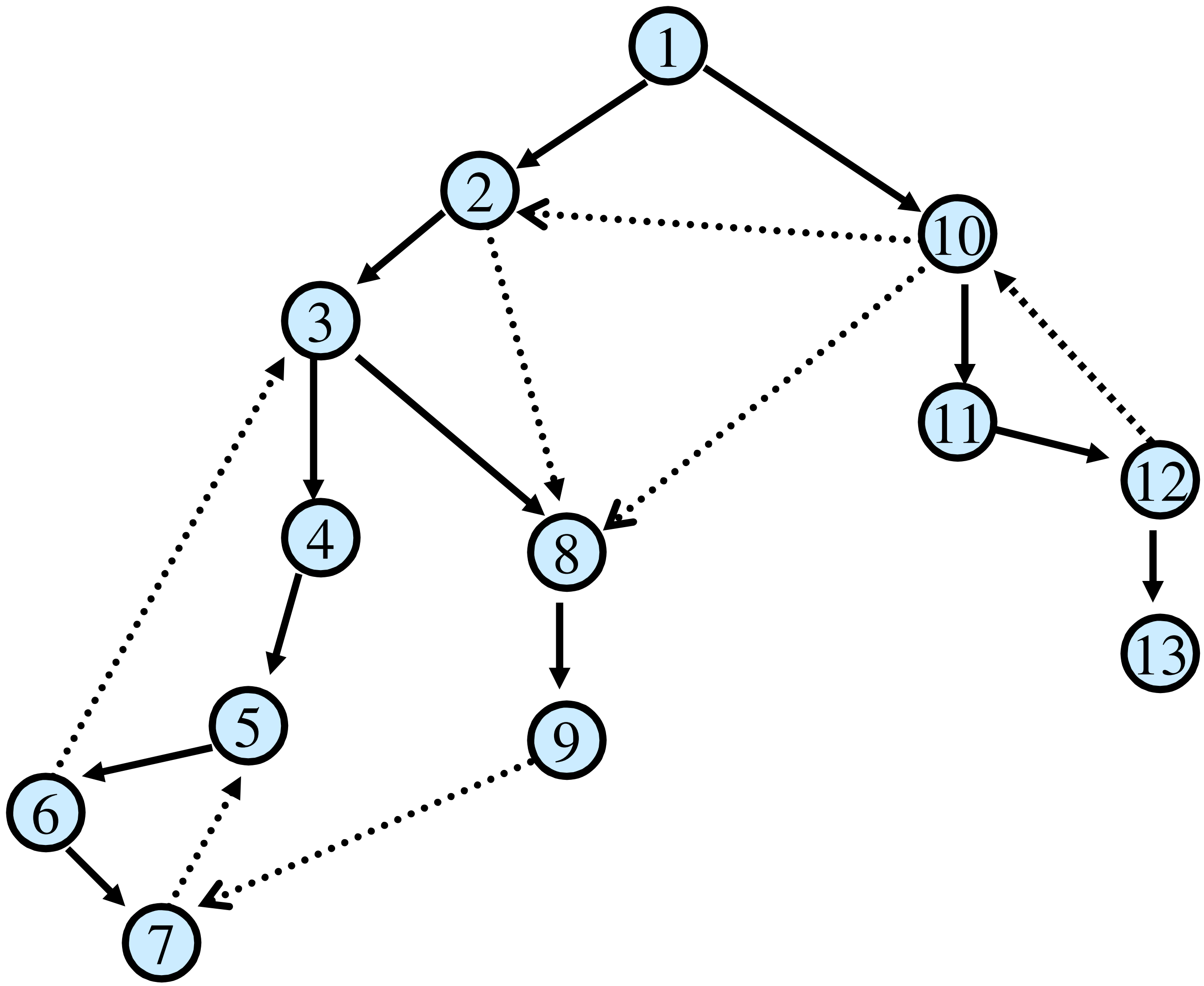
Why does it work?

u and v have a common ancestor

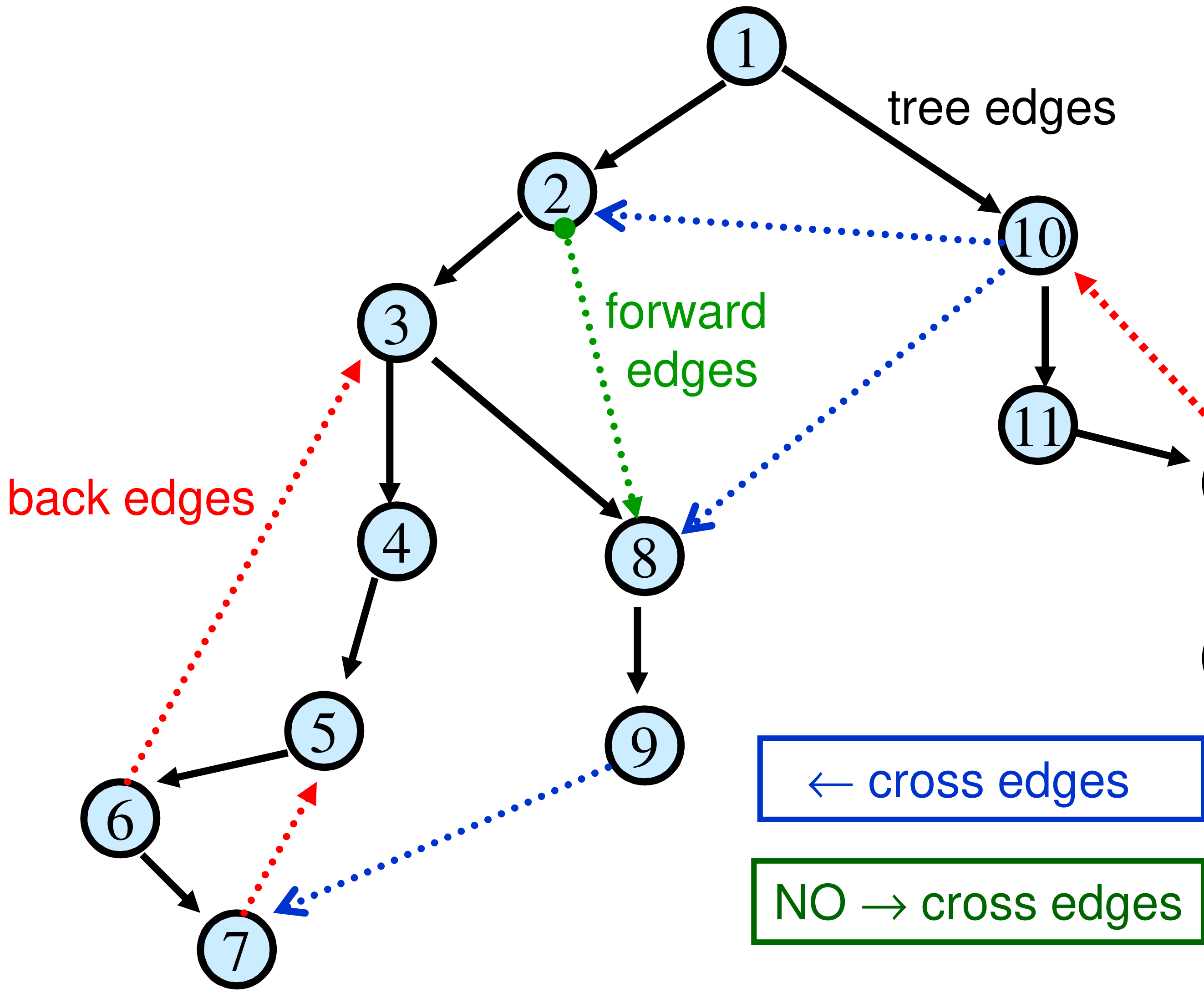
Cycle length **2(j-i)+1**



DFS(v) for a directed graph



DFS(v)


1
2
3
4
5
6
7
8
9
10
11
12
13
tree edges
forward edges
back edges
← cross edges
NO → cross edges

Properties of Directed DFS

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s

- Every cycle contains a back edge in the DFS tree

Directed Acyclic Graphs

- A directed graph $G=(V,E)$ is acyclic if it has no directed cycles

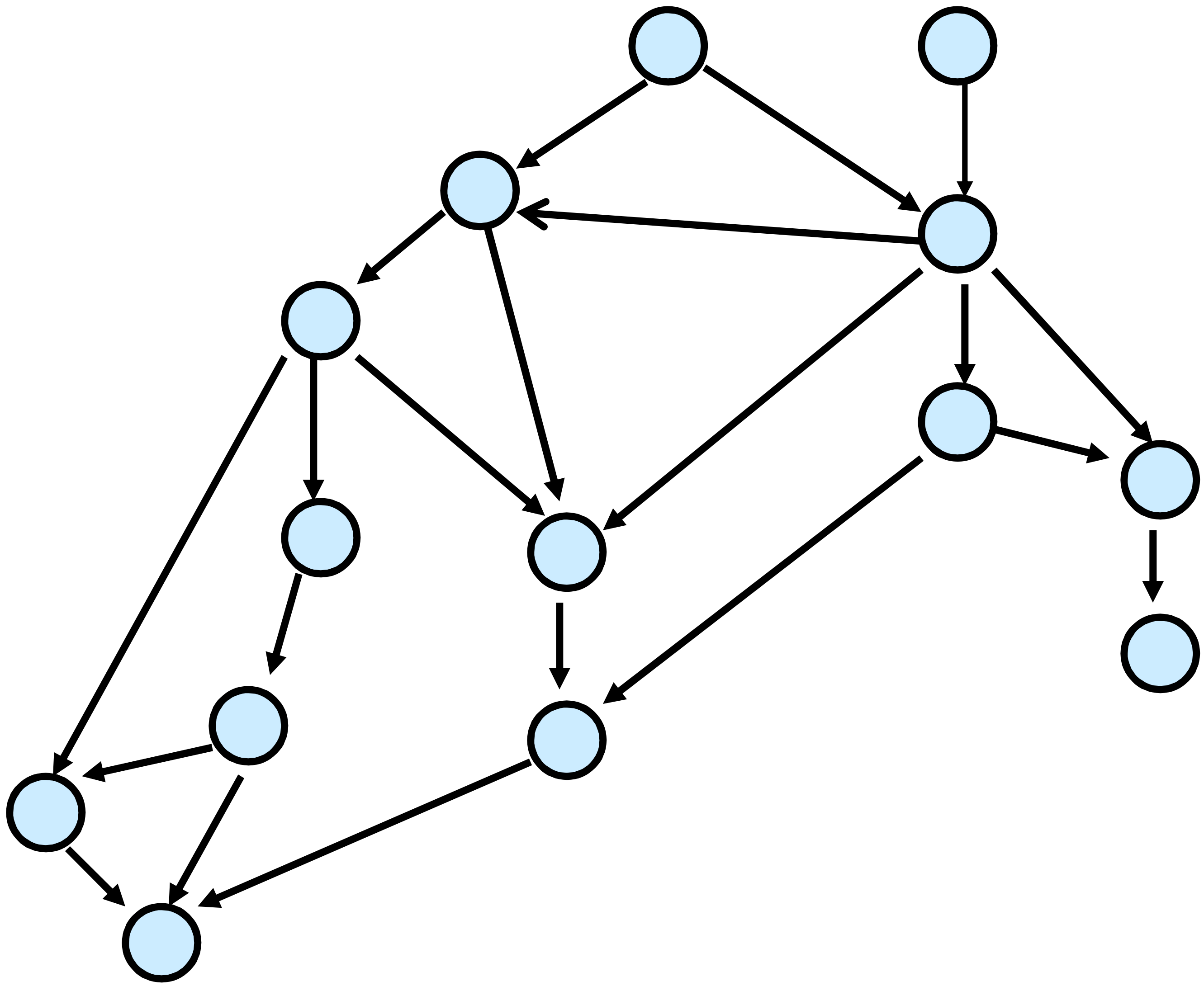
- Terminology: A directed acyclic graph is also called a DAG

Topological Sort

- **Given:** a directed acyclic graph (DAG) **G**=(**V**,**E**)
- **Output:** numbering of the vertices of **G** with distinct numbers from **1** to **n** so edges only go from lower number to higher numbered vertices

- Applications
  - nodes represent tasks
  - edges represent precedence between tasks
  - topological sort gives a sequential schedule for solving them

Directed Acyclic Graph



In-degree 0 vertices

- Every DAG has a vertex of in-degree 0
- Proof: By contradiction
  - Suppose every vertex has some incoming edge
  - Consider following procedure:

    while (true) do

      v$\leftarrow$some predecessor of **v**

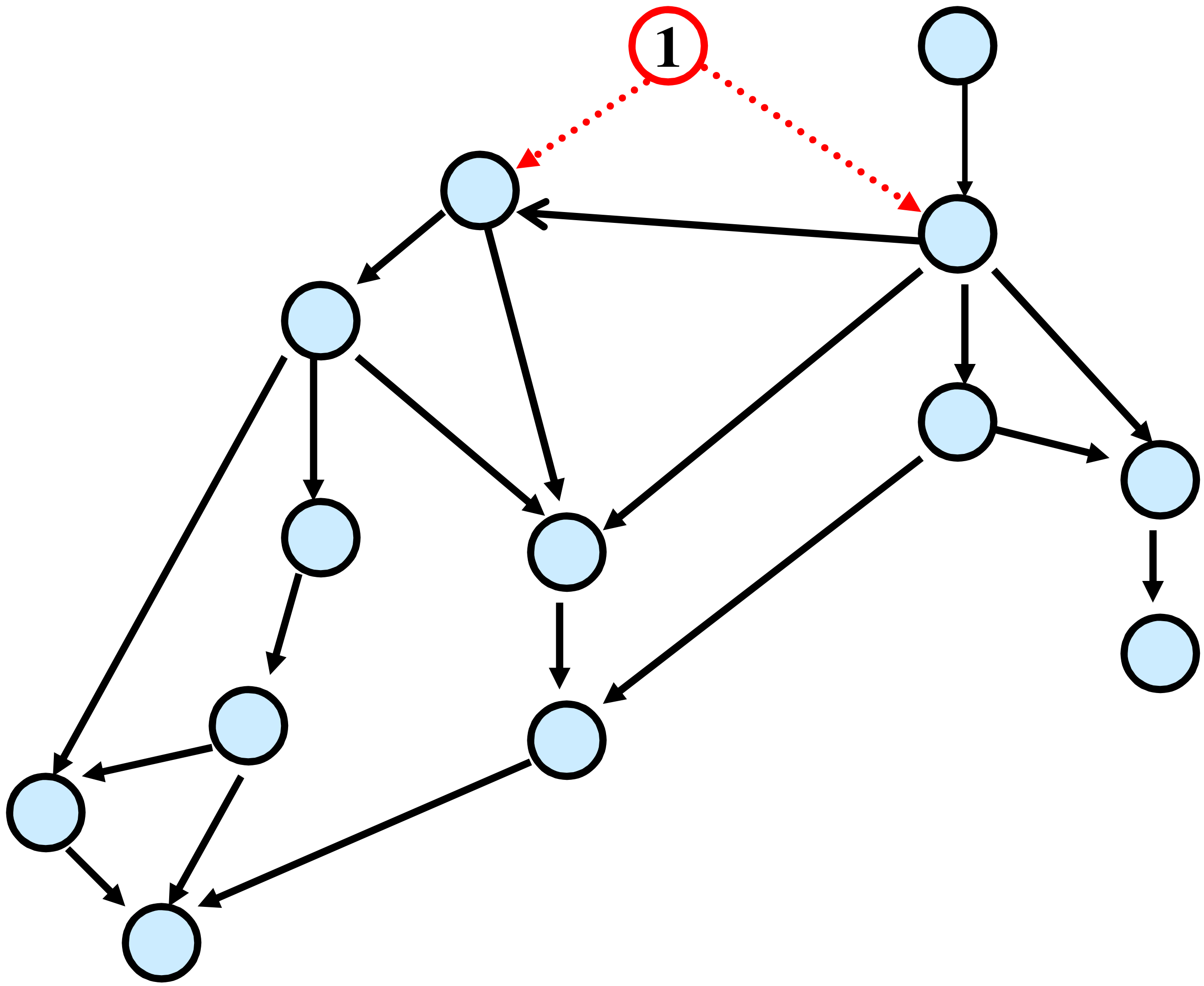
  - After **n+1** steps where **n**=|**V**| there will be a repeated vertex
    - This yields a cycle, contradicting that it is a DAG

Topological Sort

- Can do using DFS

- Alternative simpler idea:
  - Any vertex of in-degree 0 can be given number 1 to start
  - Remove it from the graph and then give a vertex of in-degree 0 number 2, etc.

Topological Sort



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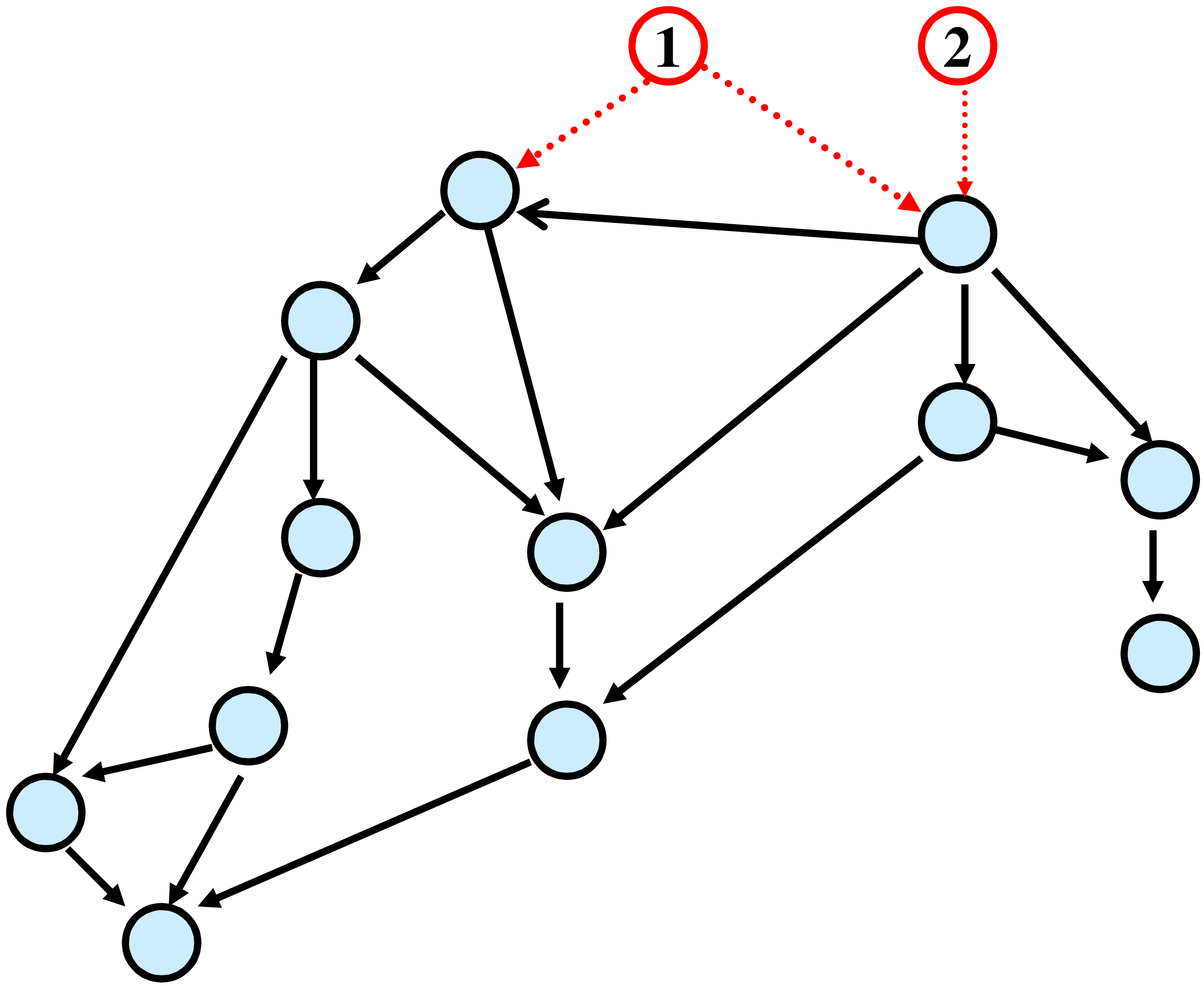

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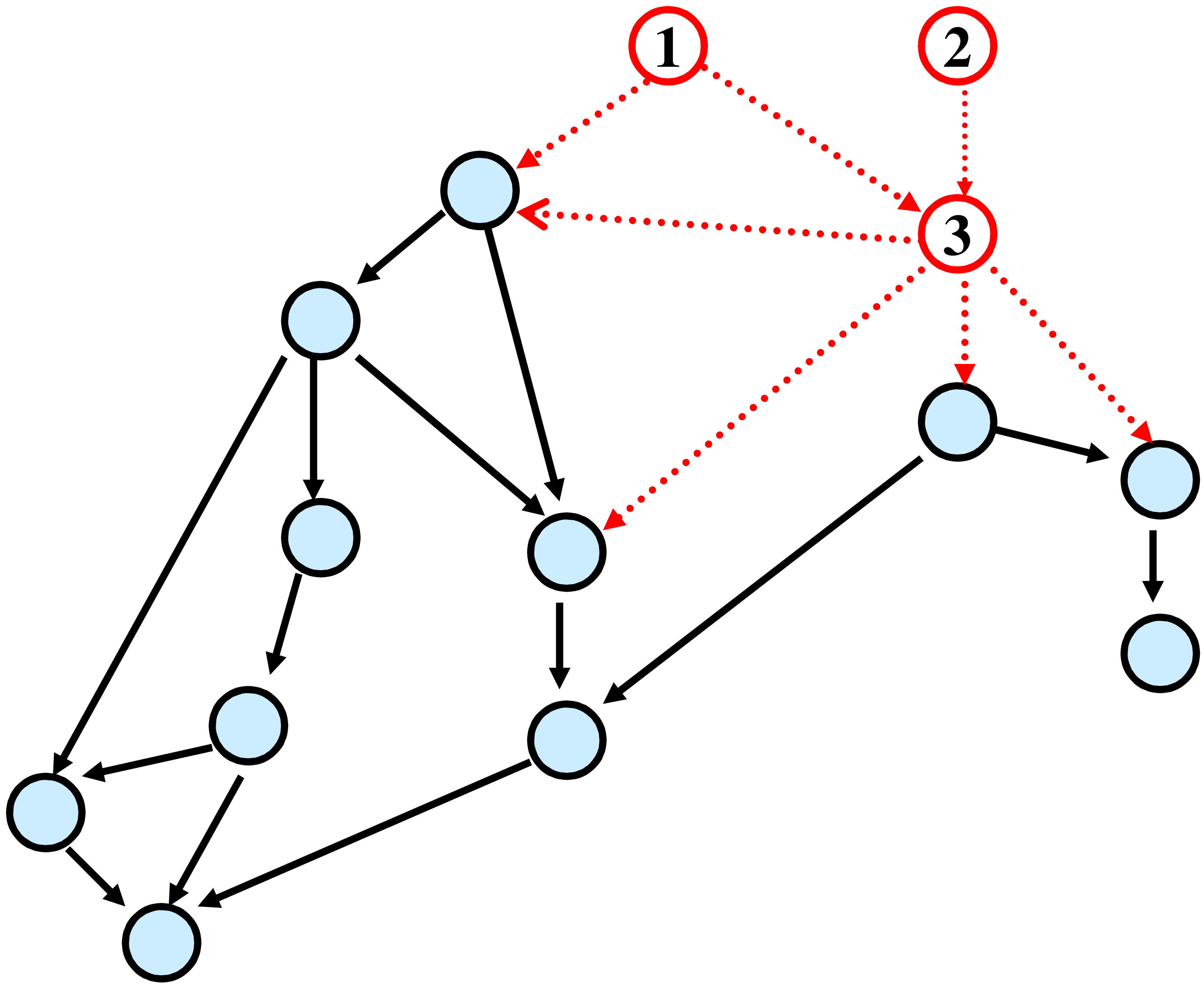

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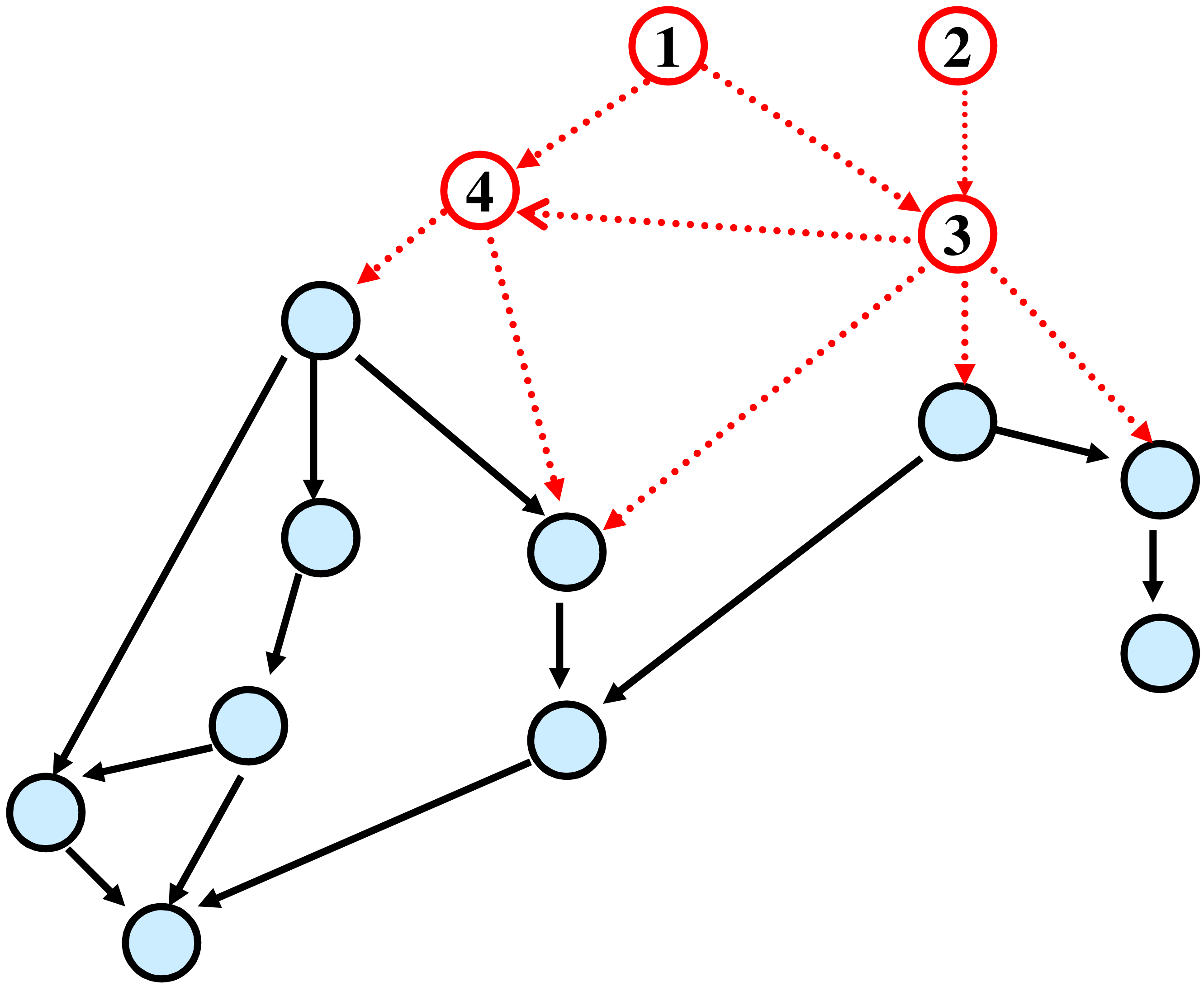

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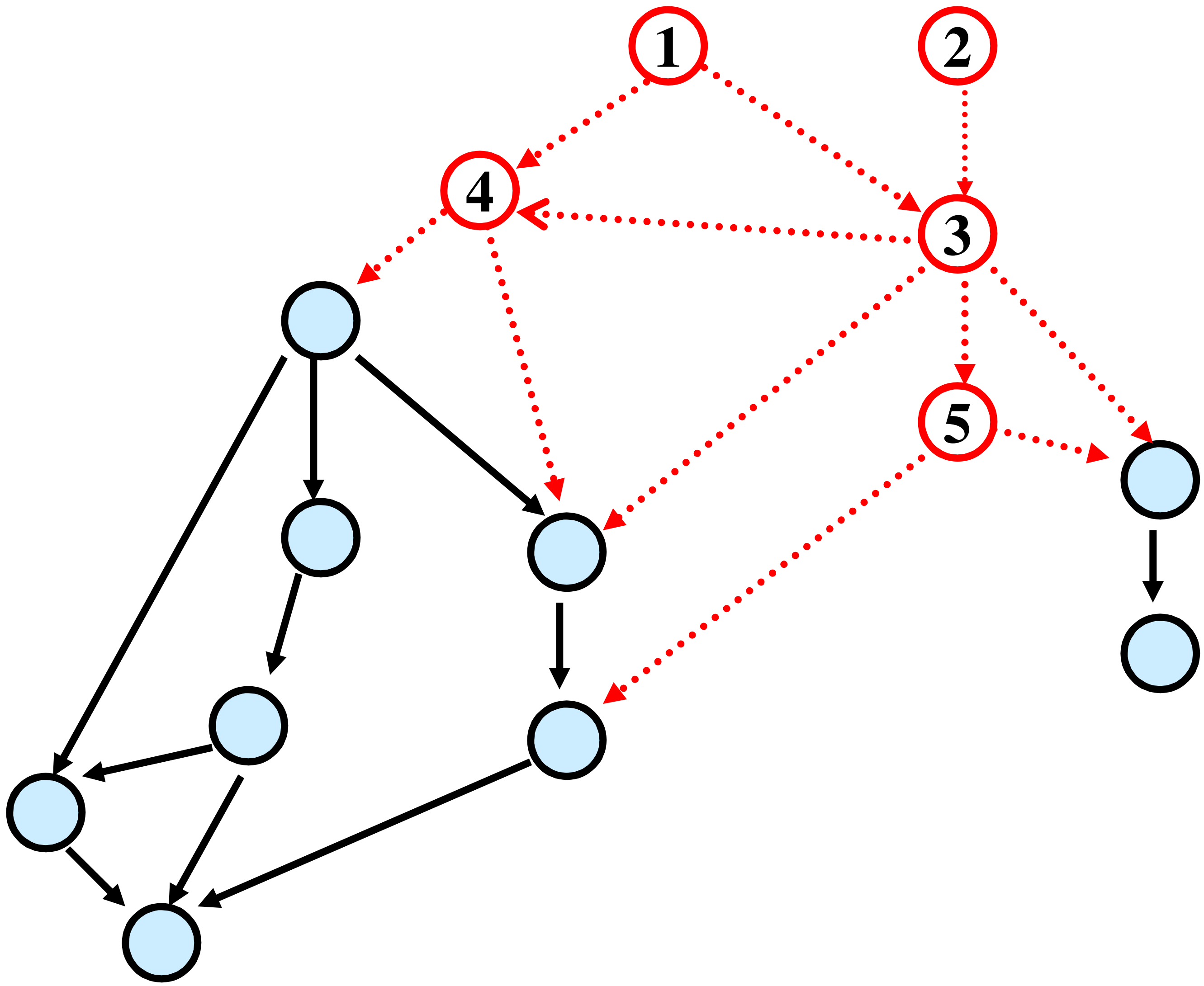

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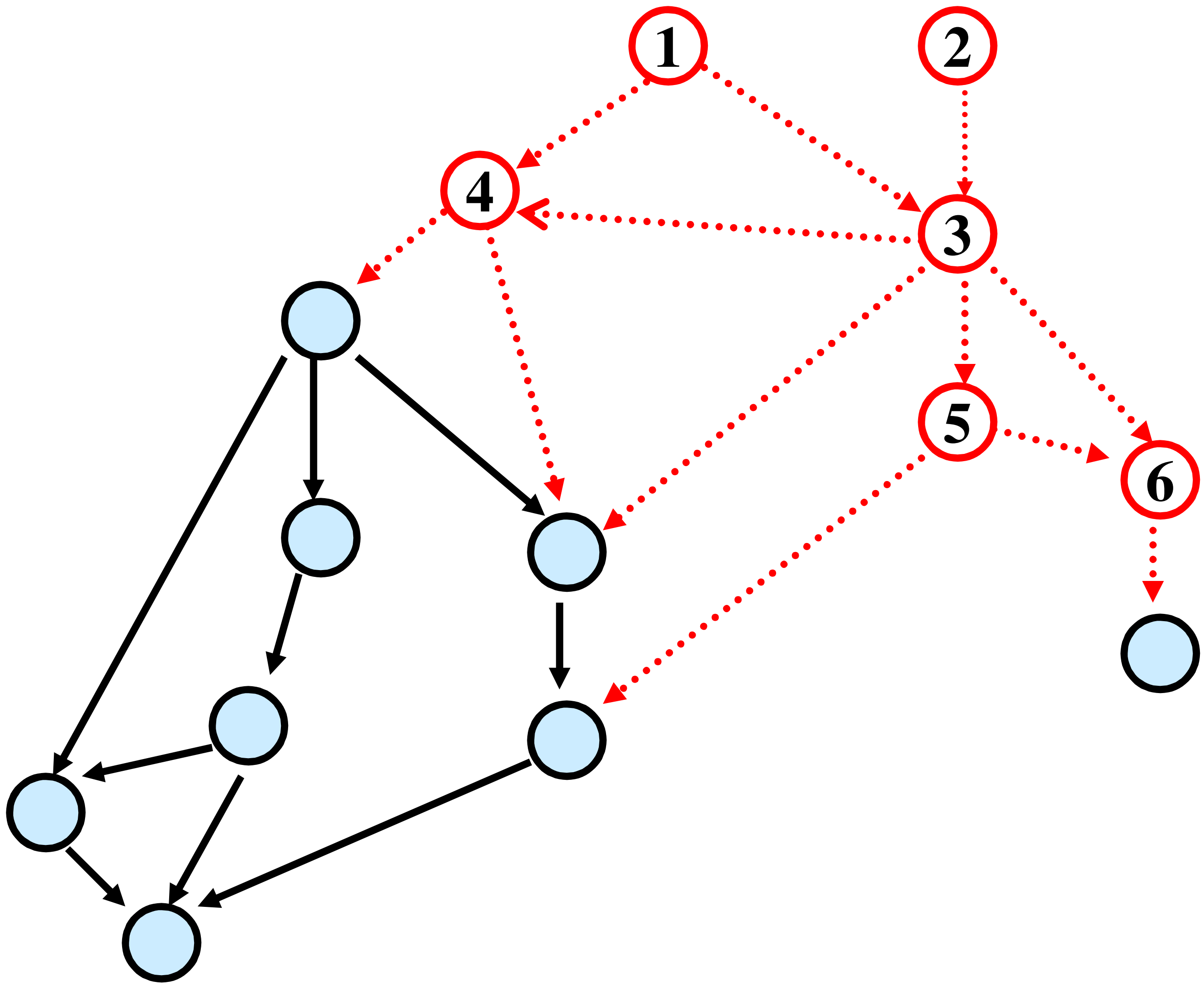

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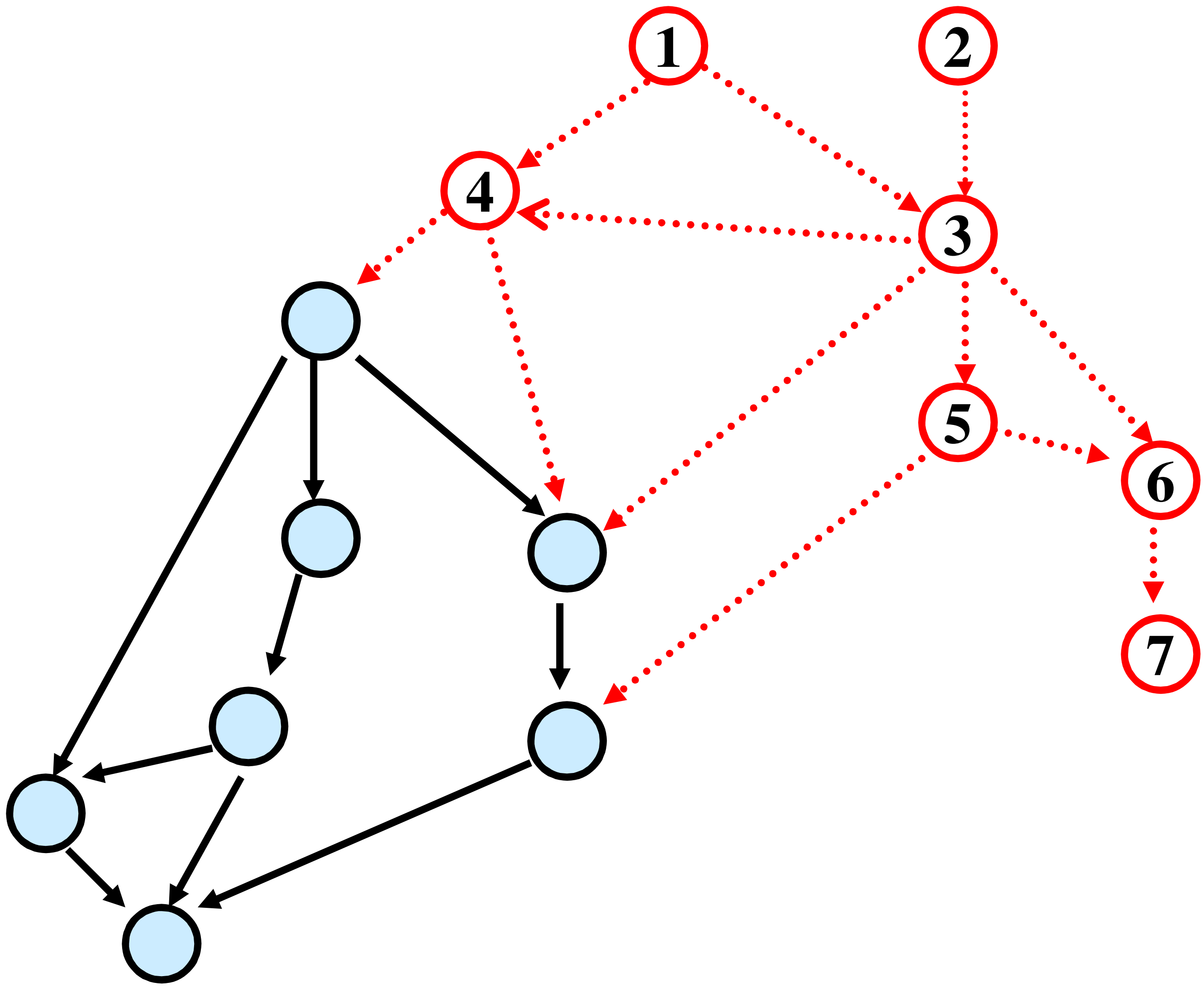

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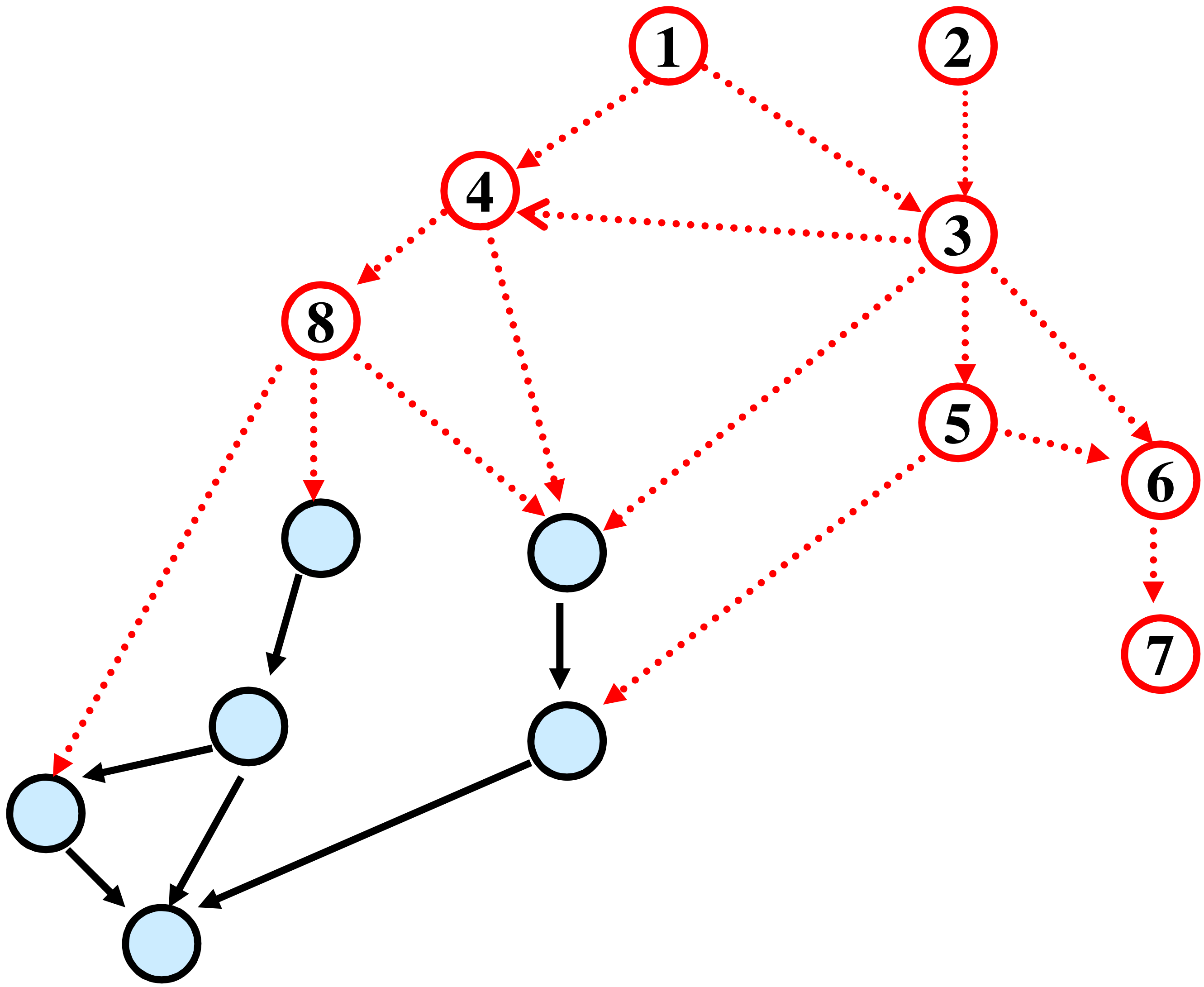

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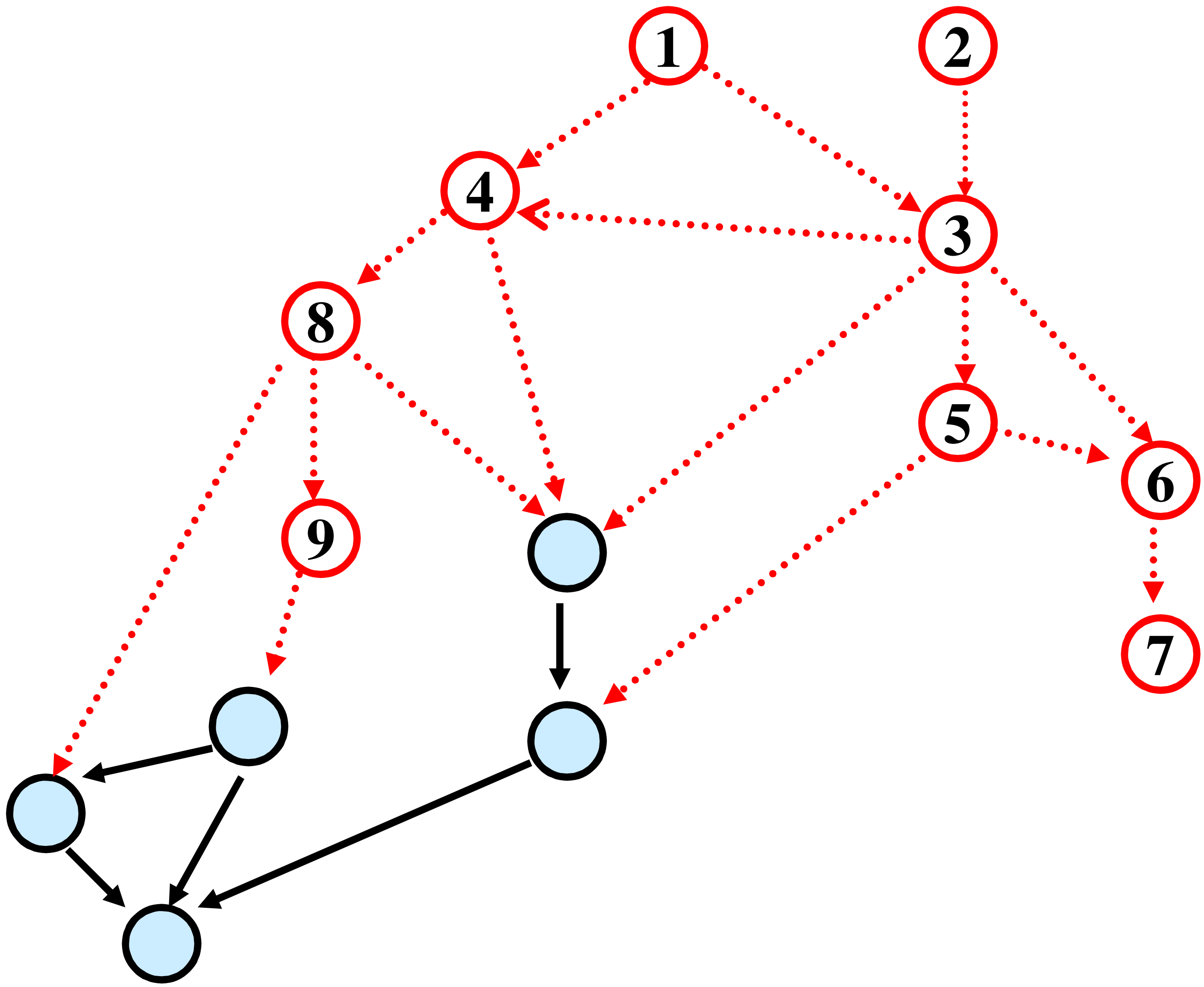

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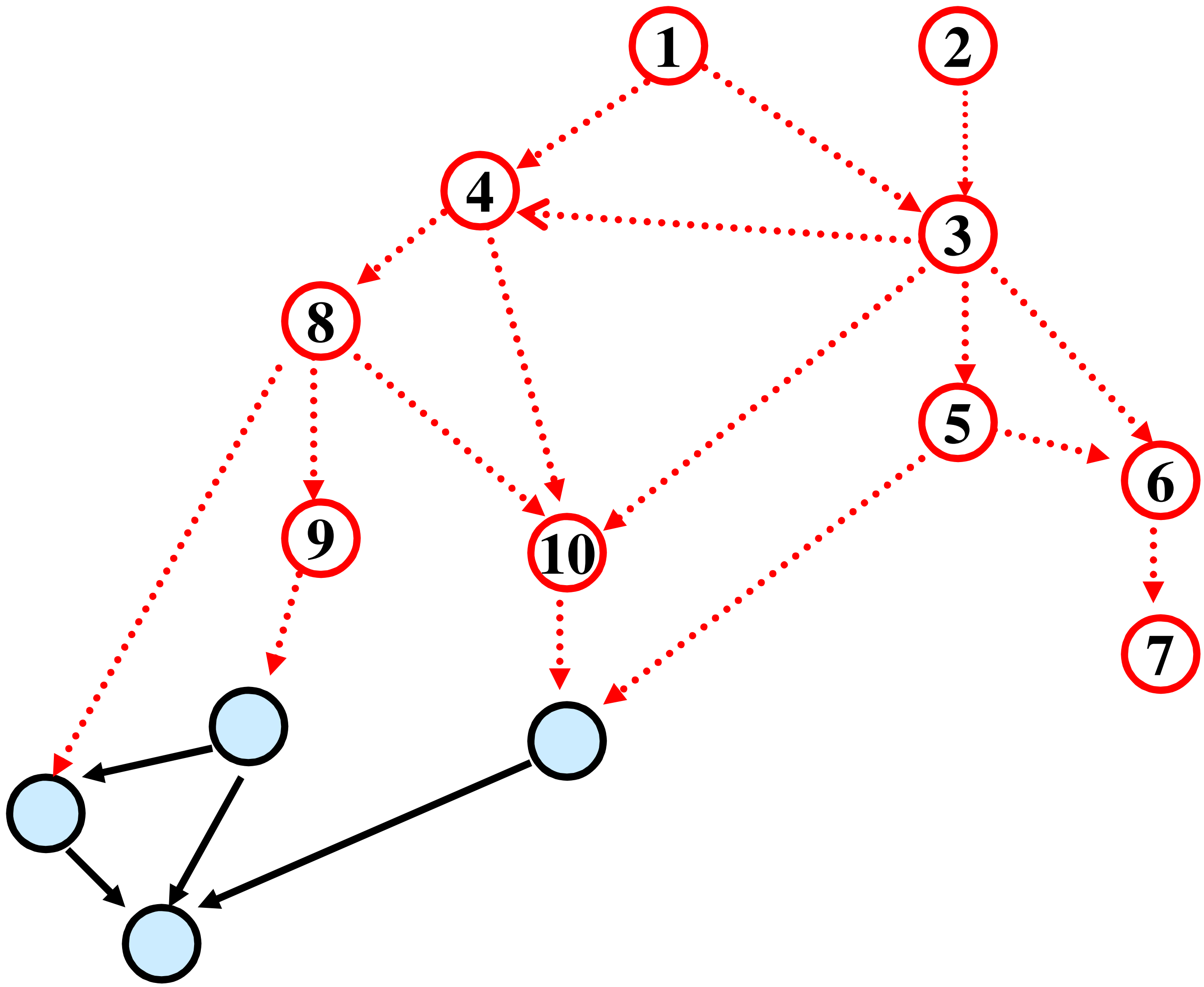

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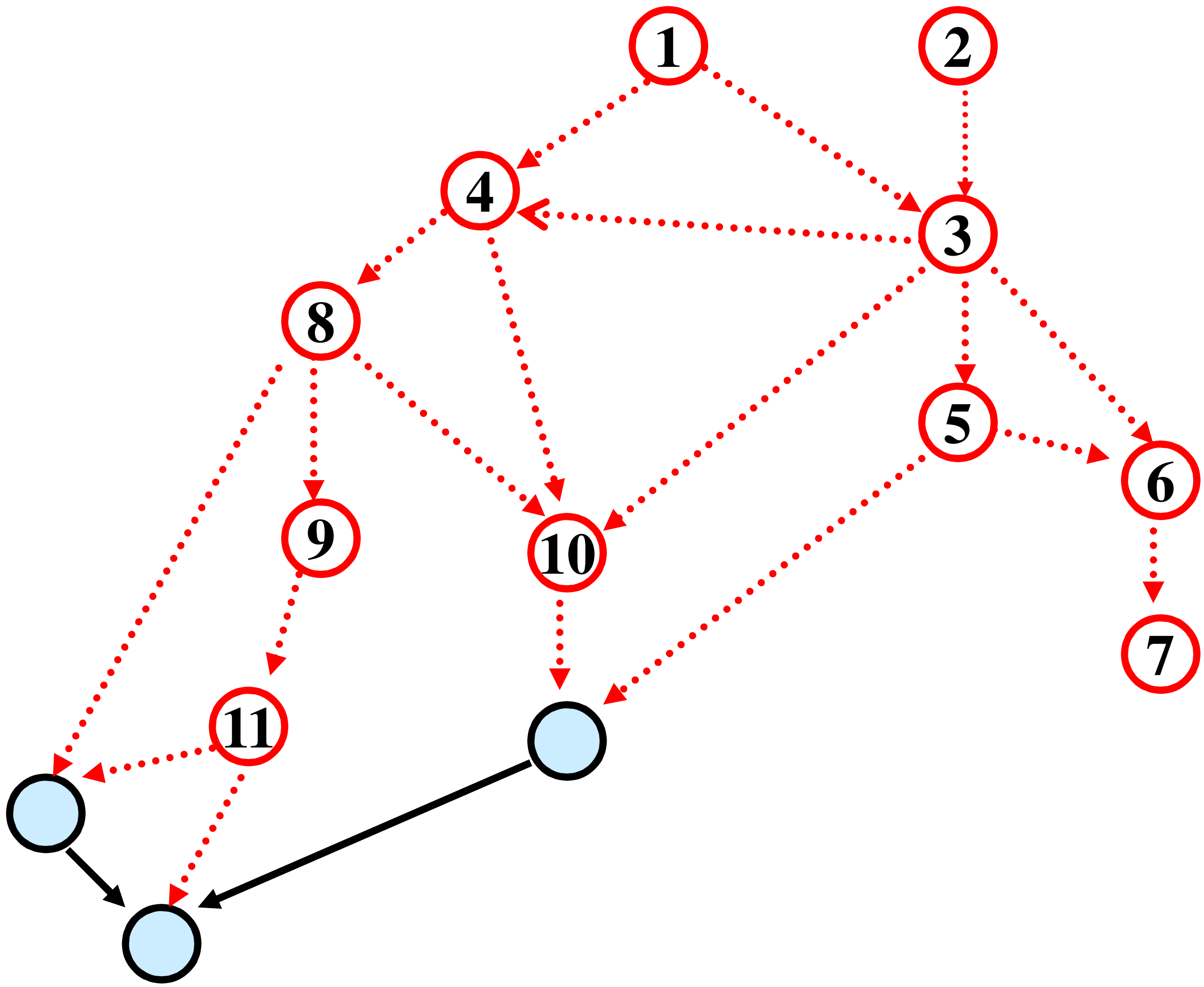

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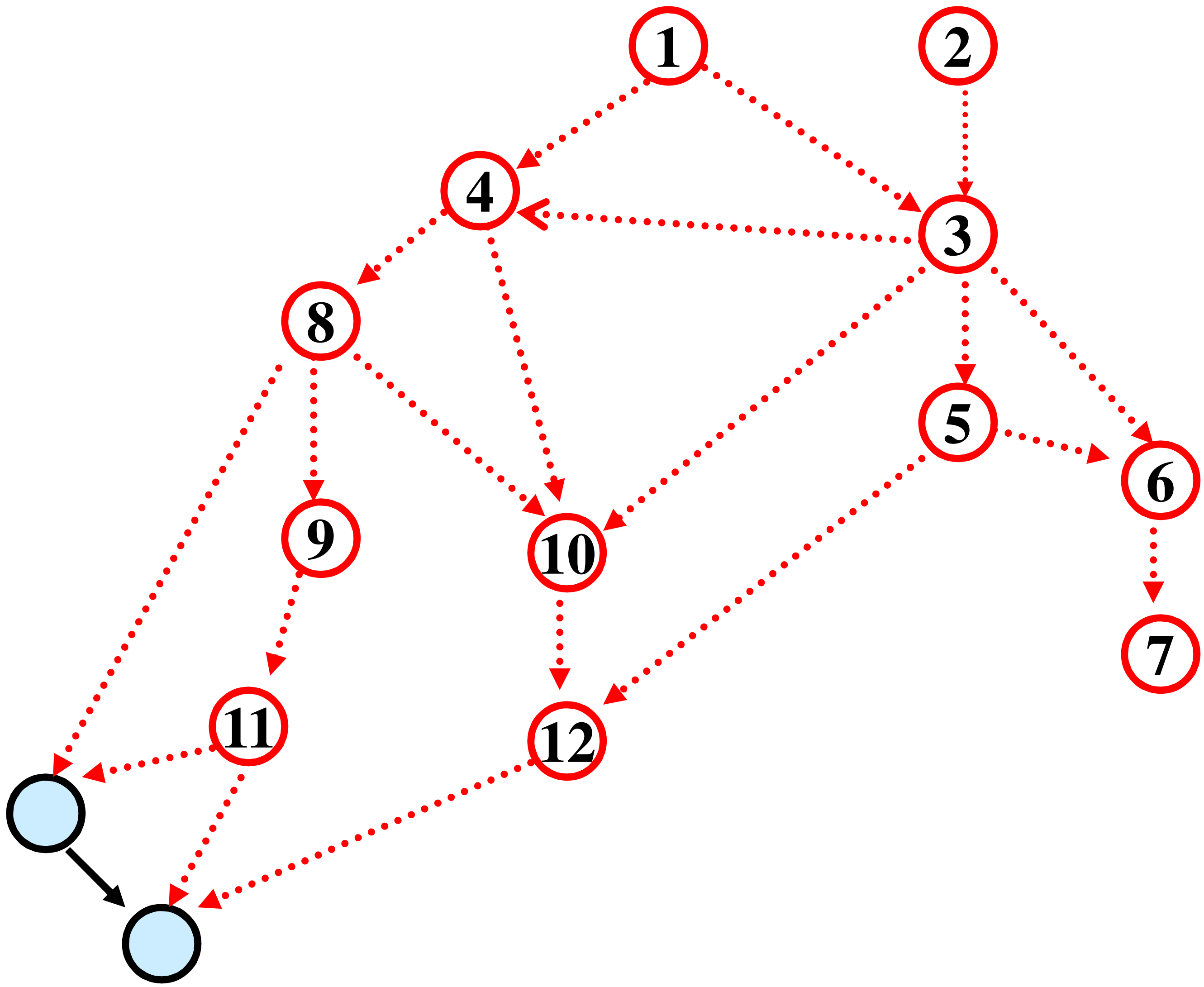

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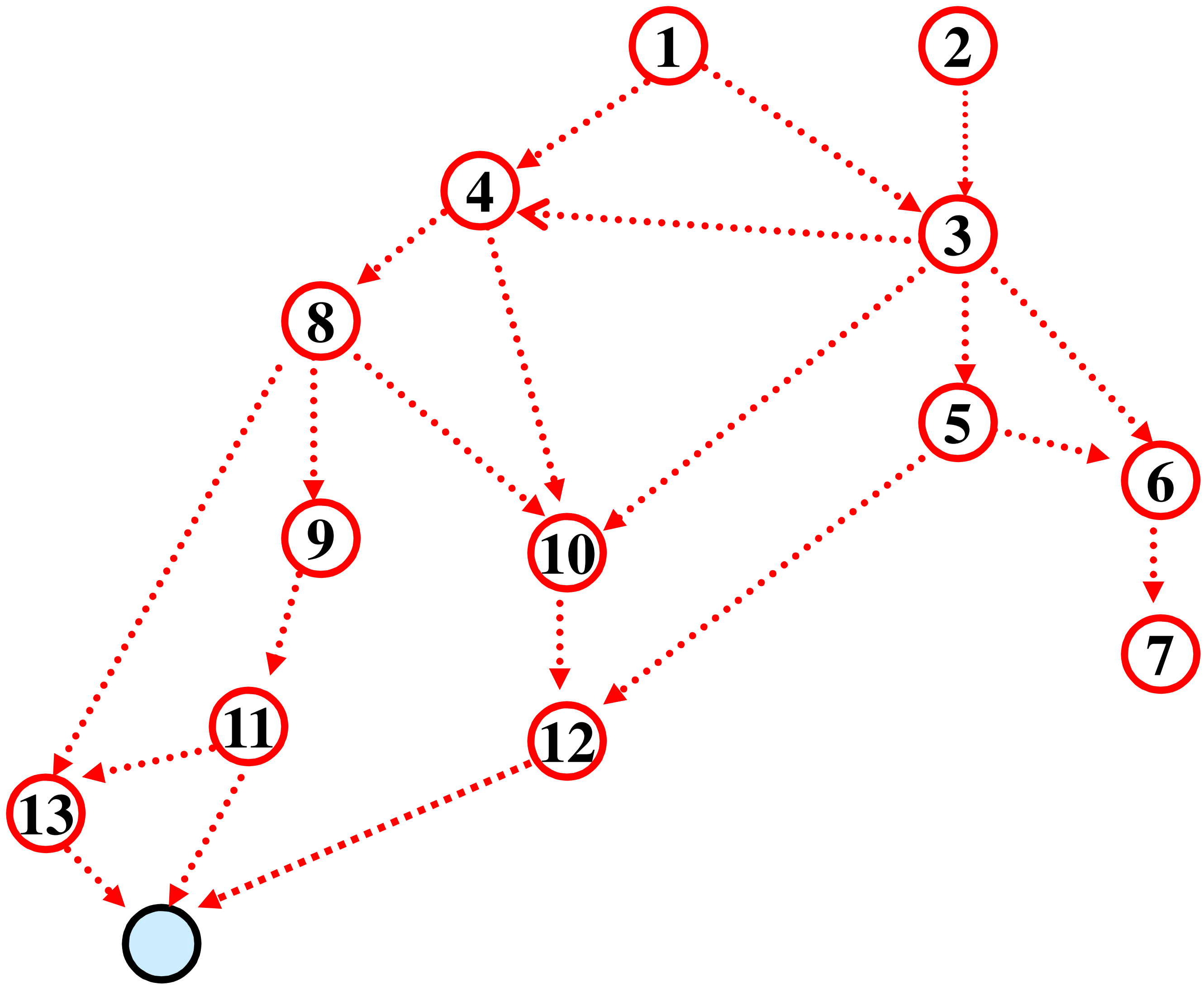

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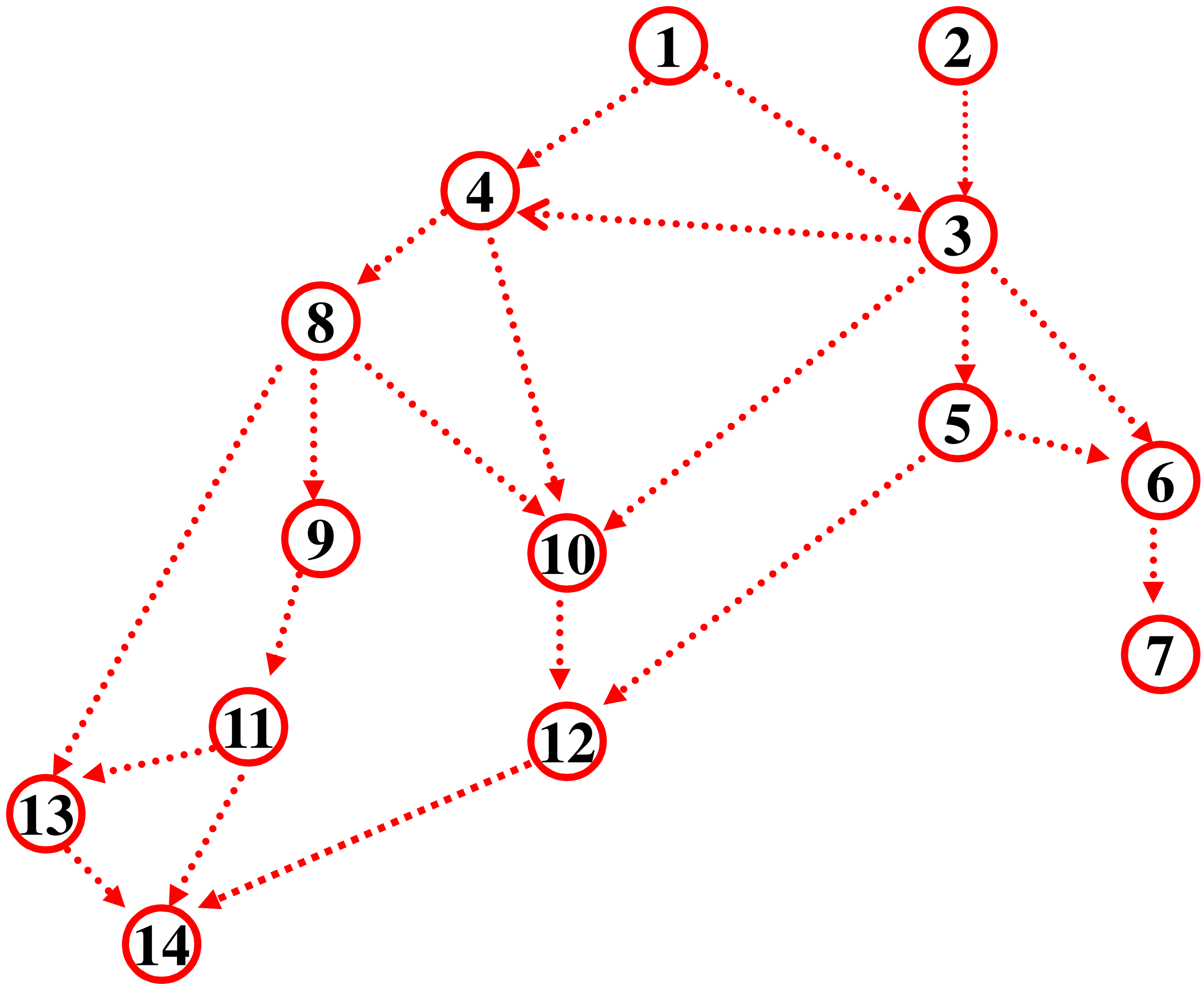

Implementing Topological Sort

- Go through all edges, computing array with in-degree for each vertex **O(m+n)**
- Maintain a queue (or stack) of vertices of in-degree **0**
- Remove any vertex in queue and number it
- When a vertex is removed, decrease in-degree of each of its neighbors by **1** and add them to the queue if their degree drops to **0**

Total cost **O(m+n)**