CSE 421: Introduction to Algorithms

Dynamic Programming

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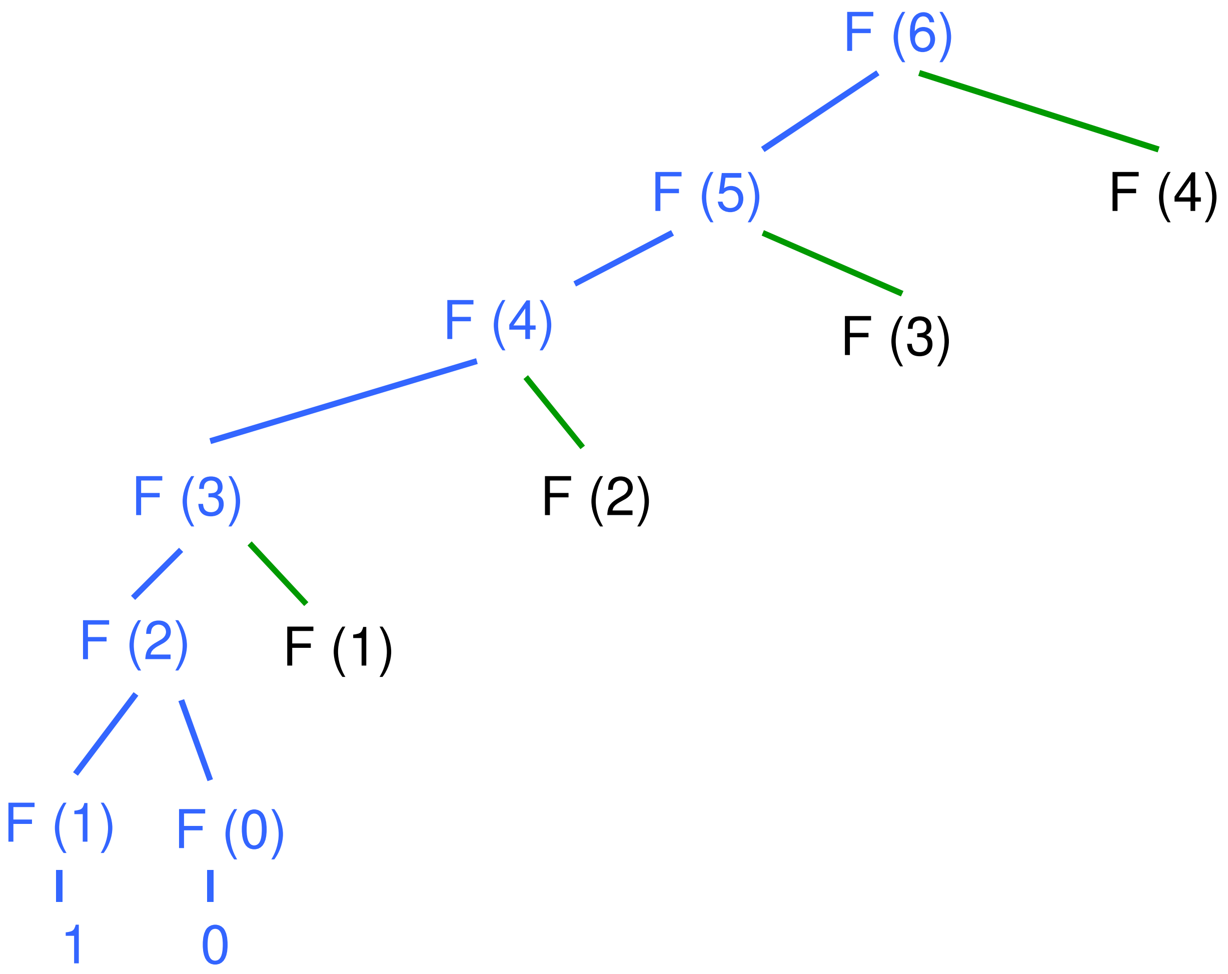
Dynamic Programming

- Dynamic Programming
  - Give a solution of a problem using smaller sub-problems where the parameters of all the possible sub-problems are determined in advance
  - Useful when the same sub-problems show up again and again in the solution

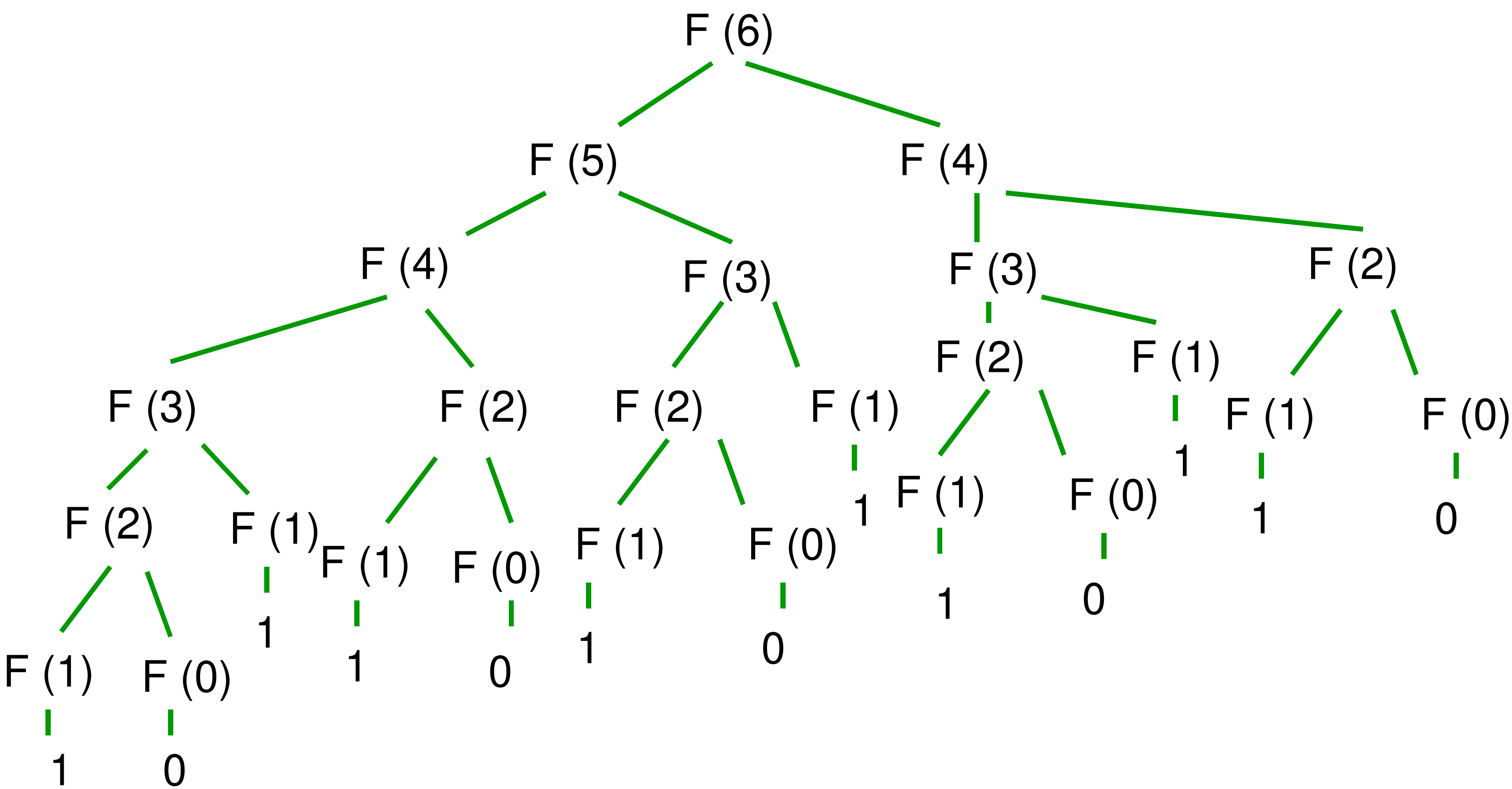
A simple case:
Computing Fibonacci Numbers

- Recall $F_n=F_{n-1}+F_{n-2}$ and $F_0=0$, $F_1=1$
- Recursive algorithm:
  - Fibo(**n**)
    if **n=0** then return(**0**)
    else if **n=1** then return(**1**)
    else return(Fibo(**n-1**)+Fibo(**n-2**))

Call tree - start



Full call tree



Memoization (Caching)

- Remember all values from previous recursive calls
- Before recursive call, test to see if value has already been computed
- Dynamic Programming
  - Convert memoized algorithm from a recursive one to an iterative one

Fibonacci
Dynamic Programming Version

- FiboDP(**n**):

```
F[0]← 0
F[1] ←1
for i=2 to n do
    F[i]←F[i-1]+F[i-2]
endfor
return(F[n])
```

Fibonacci: Space-Saving Dynamic Programming

- FiboDP(**n**):

```
prev← 0
curr←1
for i=2 to n do
    temp←curr
    curr←curr+prev
    prev←temp
endfor
return(curr)
```

Dynamic Programming

- Useful when
  - same recursive sub-problems occur repeatedly
  - Can anticipate the parameters of these recursive calls
  - The solution to whole problem can be figured out with knowing the internal details of how the sub-problems are solved
    - principle of optimality

      "Optimal solutions to the sub-problems suffice for optimal solution to the whole problem"

Three Steps to Dynamic Programming

- Formulate the answer as a recurrence relation or recursive algorithm
- Show that the number of different values of parameters in the recursive calls is “small”
  - e.g., bounded by a low-degree polynomial
  - Can use memoization
- Specify an order of evaluation for the recurrence so that you already have the partial results ready when you need them.

Weighted Interval Scheduling

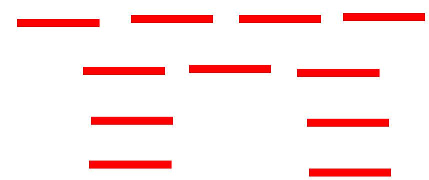
- Same problem as interval scheduling except that each request **i** also has an associated value or weight w_i
  - w_i might be
    - amount of money we get from renting out the resource for that time period
    - amount of time the resource is being used $w_i = f_i - s_i$
- Goal: Find compatible subset **S** of requests with maximum total weight

Greedy Algorithms for Weighted Interval Scheduling?

- No criterion seems to work
  - Earliest start time s_i
    - Doesn't work

  - Shortest request time f_i-s_i
    - Doesn't work

  - Fewest conflicts
    - Doesn't work



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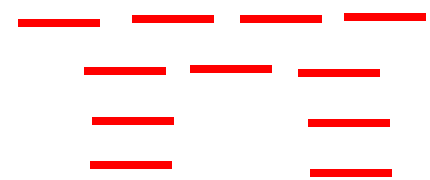

  - Earliest finish fime f_i
    - Doesn't work

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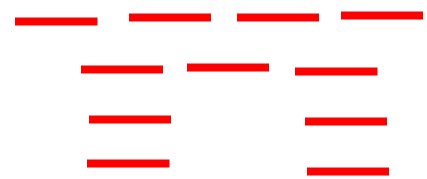

  - Earliest finish fime f_i
    - Doesn't work

  - Largest weight w_i
    - Doesn't work

Towards Dynamic Programming: Step 1 – A Recursive Algorithm

- Suppose that like ordinary interval scheduling we have first sorted the requests by finish time f_i so $f_1 \leq f_2 \leq \ldots \leq f_n$
- Say request i comes **before** request j if $i < j$

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- For any request **j** let **p(j)** be
  - the largest-numbered request before **j** that is compatible with **j**
  - or **0** if no such request exists

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- Say request **i** comes **before** request **j** if $i < j$
- For any request **j** let $p(j)$ be
  - the largest-numbered request before **j** that is compatible with **j**
  - or **0** if no such request exists
- Therefore $\{1,\ldots,p(j)\}$ is precisely the set of requests before **j** that are compatible with **j**

Towards Dynamic Programming: Step 1 – A Recursive Algorithm

- Two cases depending on whether an optimal solution **O** includes request **n**
  - If it **does** include request **n** then all other requests in **O** must be contained in **{1,…,p(n)}**

Towards Dynamic Programming: Step 1 – A Recursive Algorithm

- Two cases depending on whether an optimal solution **O** includes request **n**
  - If it **does** include request **n** then all other requests in **O** must be contained in {**1**,…,**p**(**n**)}
    - Not only that!
      - Any set of requests in {**1**,…,**p**(**n**)} will be compatible with request **n**
      - So in this case the optimal solution **O** must contain an optimal solution for {**1**,…,**p**(**n**)}
      - **"Principle of Optimality"**

Towards Dynamic Programming: Step 1 – A Recursive Algorithm

- All subproblems involve requests $\{1,..,i\}$ for some i

- For $i=1,\ldots,n$ let $OPT(i)$ be the weight of the optimal solution to the problem $\{1,\ldots,i\}$

- The two cases give

  $$OPT(n)=\max[w_n+OPT(p(n)),OPT(n-1)]$$

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- The two cases give

  $$OPT(n)=\max[w_n+OPT(p(n)),OPT(n-1)]$$

- Also
  - $n \in O$ iff $w_n+OPT(p(n))>OPT(n-1)$

Towards Dynamic Programming: Step 1 – A Recursive Algorithm

- Sort requests and compute array **p[i]** for each **i=1,…,n**

```
ComputeOpt(n)
    if n=0 then return(0)
    else
        u←ComputeOpt(p[n])
        v←ComputeOpt(n-1)
        if w_n+u>v then return(w_n+u)
                        else return(v)
    endif
```

Towards Dynamic Programming: Step 2 – Small # of parameters

- ComputeOpt(**n**) can take exponential time in the worst case
  - $\mathbf{2^n}$ calls if **p**(**i**)=**i-1** for every **i**

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- Store these answers in an array **OPT[n]** and only recompute when necessary
  - Memoization

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  - 2^n calls if **p**(**i**)=**i-1** for every **i**
- There are only **n** possible parameters to ComputeOpt
- Store these answers in an array **OPT**[**n**] and only recompute when necessary
  - Memoization
- Initialize **OPT**[**i**]=**0** for **i**=**1**,…,**n**

Dynamic Programming: Step 2 – Memoization

```
ComputeOpt(n)
    if n=0 then return(0)
    else
        u←MComputeOpt(p[n])
        v←MComputeOpt(n-1)
        if w_n+u>v then
            return(w_n+u)
        else return(v)
    endif
```

```
MComputeOpt(n)
    if OPT[n]=0 then
        v←ComputeOpt(n)
        OPT[n]←v
        return(v)
    else
        return(OPT[n])
    endif
```

Dynamic Programming Step 3: Iterative Solution

- The recursive calls for parameter **n** have parameter values **i** that are $<$ **n**

```
IterativeComputeOpt(n)
   array OPT[0..n]
   OPT[0]←0
   for i=1 to n
        if w_i+OPT[p[i]] >OPT[i-1] then
                OPT[i] ←w_i+OPT[p[i]]
        else
                OPT[i] ←OPT[i-1]
        endif
   endfor
```

Producing the Solution

```
IterativeComputeOptSolution(n)
  array OPT[0..n], Used[1..n]
  OPT[0]←0
  for i=1 to n
    if w_i+OPT[p[i]] >OPT[i-1] then
      OPT[i] ←w_i+OPT[p[i]]
      Used[i]←1
    else
      OPT[i] ← OPT[i-1]
      Used[i] ←0
    endif
  endfor
```

```
i←n
S←∅
while i> 0 do
    if Used[i]=1 then
        S←S ∪ {i}
        i←p[i]
    else
        i←i-1
    endif
endwhile
```

Example

	1	2	3	4	5	6	7	8	9
s_i	4	2	6	8	11	15	11	12	18
f_i	7	9	10	13	14	17	18	19	20
w_i	3	7	4	5	3	2	7	7	2
p[i]									
OPT[i]									
Used[i]									

Example

	1	2	3	4	5	6	7	8	9
s_i	4	2	6	8	11	15	11	12	18
f_i	7	9	10	13	14	17	18	19	20
w_i	3	7	4	5	3	2	7	7	2
p[i]	0	0	0	1	3	5	3	3	7
OPT[i]									
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w_i	3	7	4	5	3	2	7	7	2
p[i]	0	0	0	1	3	5	3	3	7
OPT[i]	3	7	7	8	10	12	14	14	16
Used[i]	1	1	0	1	1	1	1	0	1

Example

	1	2	3	4	5	6	7	8	9
s_i	4	2	6	8	11	15	11	12	18
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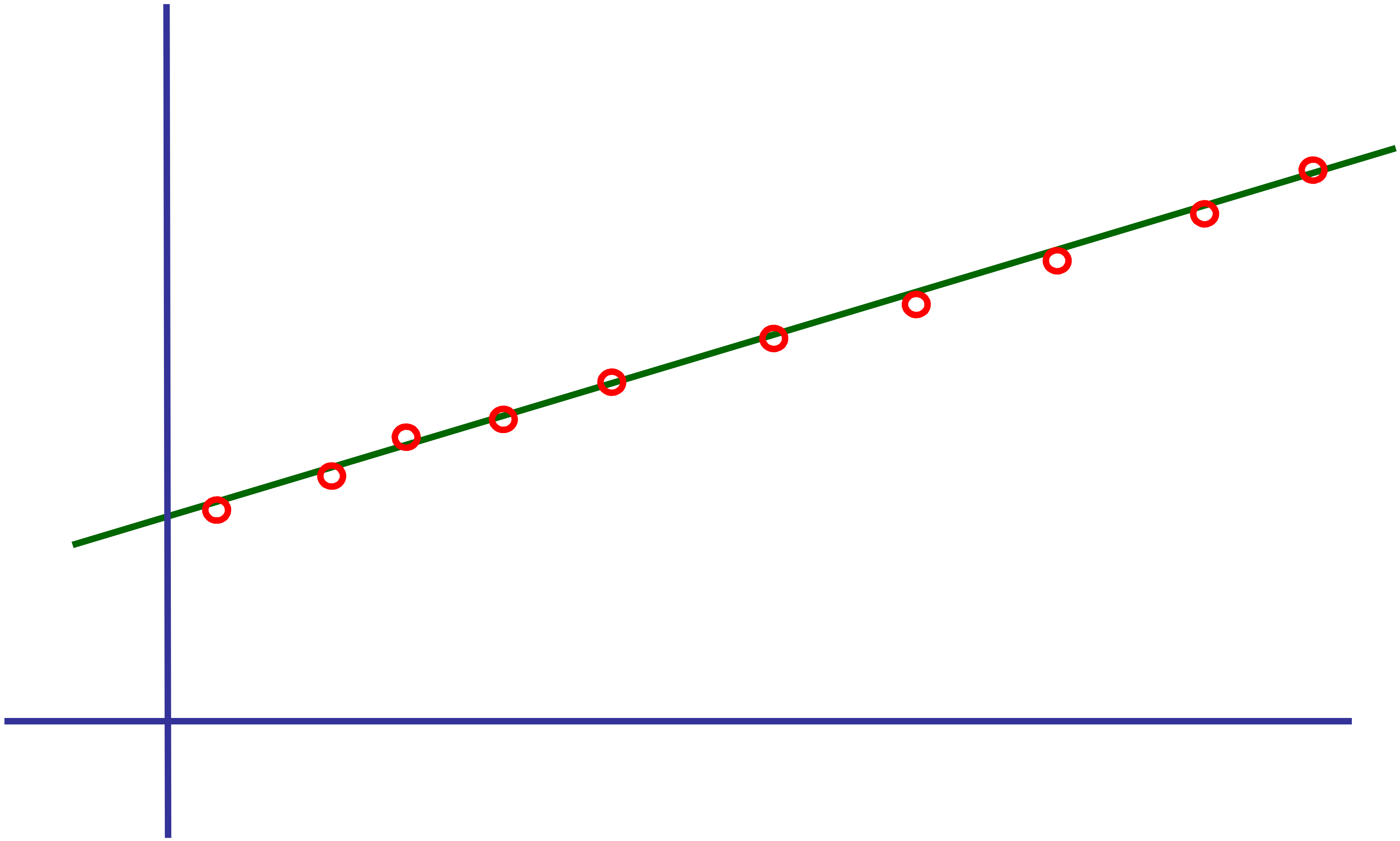
p[i]	0	0	0	1	3	5	3	3	7
OPT[i]	3	7	7	8	10	12	14	14	16
Used[i]	1	1	0	1	1	1	1	0	1

S={9,7,2}

Segmented Least Squares

- Least Squares
  - Given a set $\mathbf{P}$ of $\mathbf{n}$ points in the plane $\mathbf{p}_1=(\mathbf{x}_1,\mathbf{y}_1),\ldots,\mathbf{p}_n=(\mathbf{x}_n,\mathbf{y}_n)$ with $\mathbf{x}_1<\ldots<\mathbf{x}_n$ determine a line $\mathbf{L}$ given by $\mathbf{y}=\mathbf{ax}+\mathbf{b}$ that optimizes the total 'squared error'
    - $\text{Error}(\mathbf{L},\mathbf{P})=\sum_i(\mathbf{y}-\mathbf{ax}_i-\mathbf{b})^2$
  - A classic problem in statistics
  - Optimal solution is known (see text)
    - Call this line($\mathbf{P}$) and its error error($\mathbf{P}$)

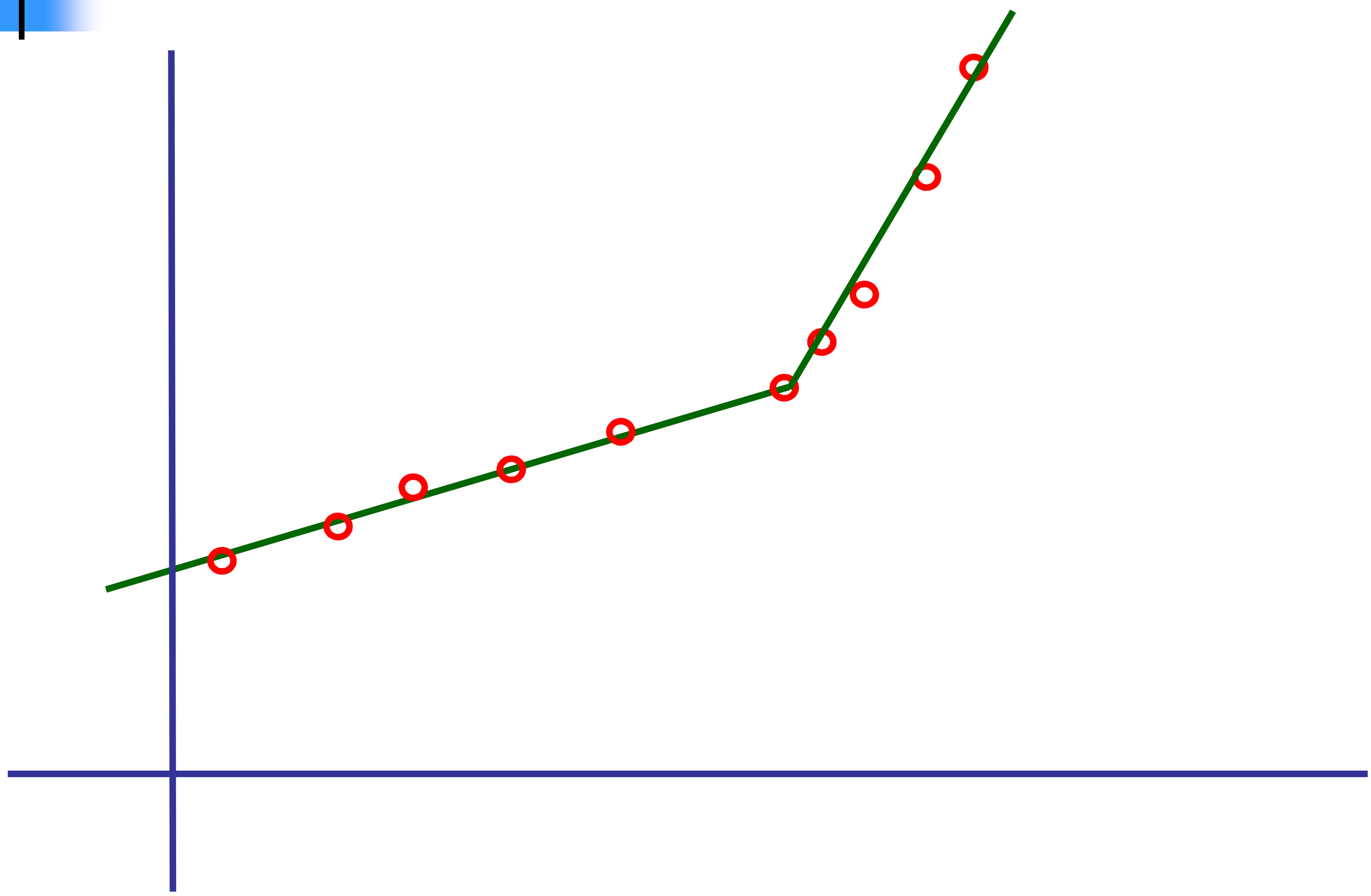
Least Squares



Segmented Least Squares

- What if data seems to follow a piece-wise linear model?

Segmented Least Squares



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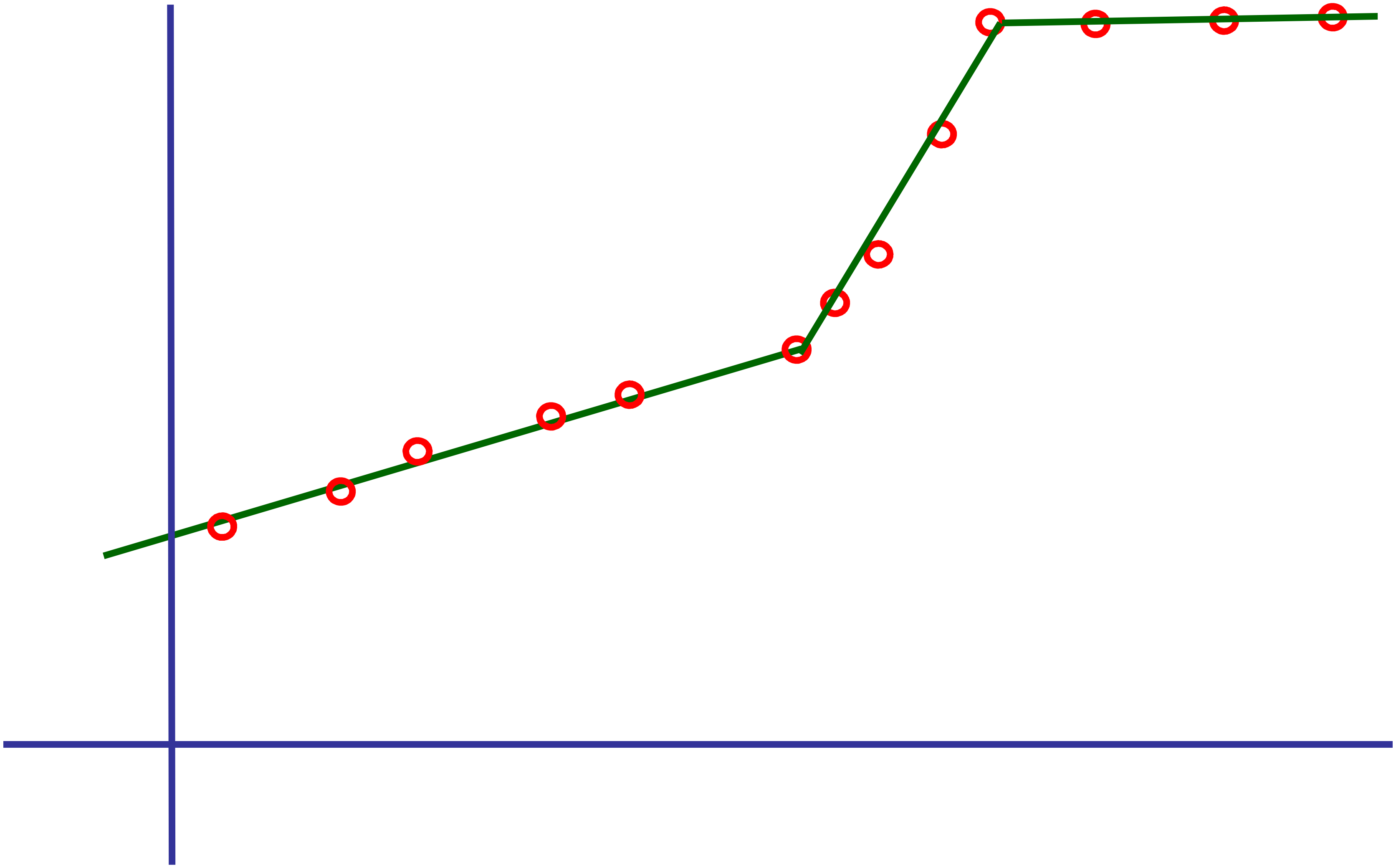

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- If we chose **n-1** pieces we could fit with **0** error
  - Not a fair measure of data fit

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- Add a penalty of **C** times the number of pieces to the error to get a total penalty

Segmented Least Squares

- What if data seems to follow a piece-wise linear model?
- Number of pieces to choose is not obvious
- If we chose **n-1** pieces we could fit with **0** error
  - Not a fair measure of data fit
- Add a penalty of **C** times the number of pieces to the error to get a total penalty
- How do we compute a solution with the smallest possible total penalty?

Segmented Least Squares

- Recursive idea
  - If we knew the point p_j where the **last** line segment began then we could solve the problem optimally for points $p_1,\ldots,p_j$ and combine that with the last segment to get a global optimal solution

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    - Let OPT($\mathbf{j}$) be the optimal penalty for points $\{\mathbf{p_1,\ldots,p_j}\}$

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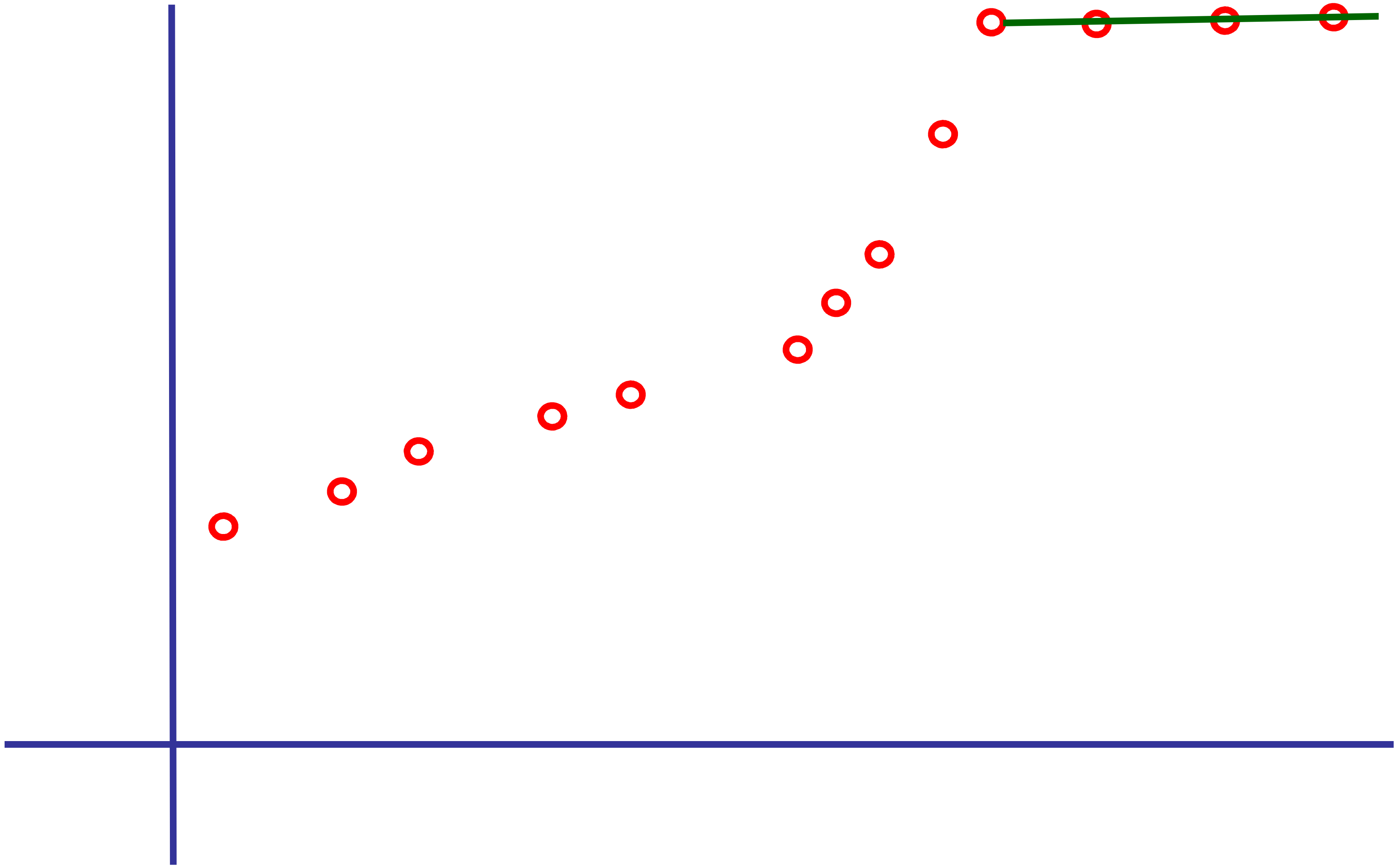
  - If we knew the point $\mathbf{p_j}$ where the **last** line segment began then we could solve the problem optimally for points $\mathbf{p_1},\ldots,\mathbf{p_j}$ and combine that with the last segment to get a global optimal solution
    - Let OPT($\mathbf{j}$) be the optimal penalty for points $\{\mathbf{p_1},\ldots,\mathbf{p_j}\}$
    - Total penalty for this solution would be

$$\text{Error}(\{\mathbf{p_j},\ldots,\mathbf{p_n}\}) + \mathbf{C} + \text{OPT}(\mathbf{j\text{-}1})$$

Segmented Least Squares



Segmented Least Squares

- Recursive idea
  - We don't know which point is p_j
    - But we do know that $1 \leq j \leq n$
    - The optimal choice will simply be the best among these possibilities

Segmented Least Squares

- Recursive idea
  - We don't know which point is $\mathbf{p_j}$
    - But we do know that $\mathbf{1 \le j \le n}$
    - The optimal choice will simply be the best among these possibilities
  - Therefore

$$\mathrm{OPT}(\mathbf{n}) = \min_{\mathbf{1 \le j \le n}} \left\{ \mathrm{Error}(\{\mathbf{p_j}, \ldots, \mathbf{p_n}\}) + \mathbf{C} + \mathrm{OPT}(\mathbf{j-1}) \right\}$$

Dynamic Programming Solution

```
SegmentedLeastSquares(n)
  array OPT[0..n]

  OPT[0]←0
  for i=1 to n
    OPT[i]←Error{(p_1,…,p_i)}+C

    for j=2 to i-1
         e←Error{(p_j,…,p_i)}+C+OPT[j-1]
         if e <OPT[i] then
            OPT[i] ←e

         endif
    endfor
  endfor
  return(OPT[n])
```

Dynamic Programming Solution

```
SegmentedLeastSquares(n)
  array OPT[0..n]
  array Begin[1..n]
  OPT[0]←0
  for i=1 to n
     OPT[i]←Error{(p_1,…,p_i)}+C
     Begin[i]←1
     for j=2 to i-1
        e←Error{(p_j,…,p_i)}+C+OPT[j-1]
        if e <OPT[i] then
           OPT[i] ←e
           Begin[i]←j
        endif
     endfor
  endfor
  return(OPT[n])
```

Dynamic Programming Solution

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SegmentedLeastSquares(n)
  array OPT[0..n]
  array Begin[1..n]
  OPT[0]←0
  for i=1 to n
     OPT[i]←Error{(p_1,...,p_i)}+C
     Begin[i]←1
     for j=2 to i-1
          e←Error{(p_j,…,p_i)}+C+OPT[j-1]
          if e <OPT[i] then
               OPT[i] ←e
               Begin[i]←j
          endif
     endfor
  endfor
  return(OPT[n])
```

```
FindSegments
i←n
S←∅
while i> 1 do
     compute Line({p_Begin[i],…,p_i})
     output (p_Begin[i],p_i), Line
     i←Begin[i]
endwhile
```

Knapsack (Subset-Sum) Problem

- Given:
  - integer **W** (knapsack size)
  - **n** object sizes $x_1, x_2, \ldots, x_n$
- Find:
  - Subset **S** of $\{1, \ldots, n\}$ such that $\sum_{i \in S} x_i \le W$ but $\sum_{i \in S} x_i$ is as large as possible

Recursive Algorithm

- Let $K(n,W)$ denote the problem to solve for W and $x_1, x_2, \dots, x_n$
- For $n>0$,
  - The optimal solution for $K(n,W)$ is the better of the optimal solution for either

    $$K(n-1,W) \text{ or } x_n + K(n-1, W-x_n)$$

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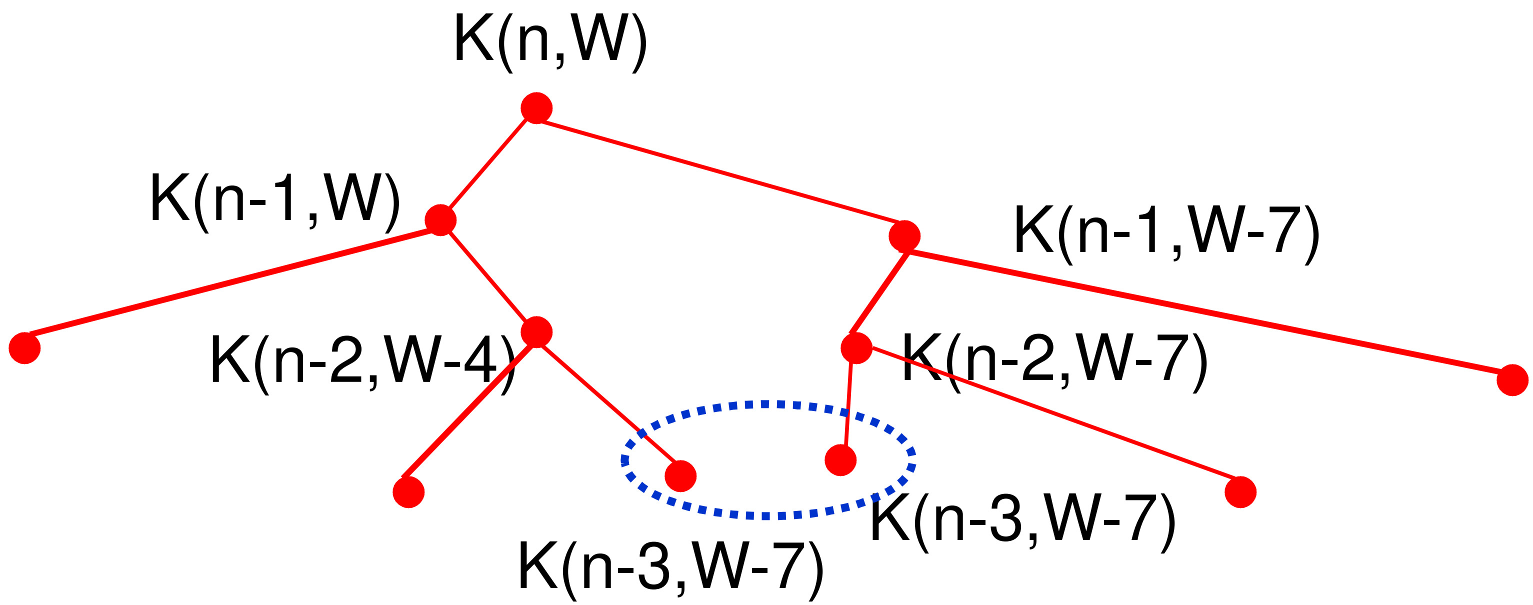
- For $n>0$,
  - The optimal solution for $K(n,W)$ is the better of the optimal solution for either

    $$K(n-1,W) \text{ or } x_n+K(n-1,W-x_n)$$
- For $n=0$
  - $K(0,W)$ has a trivial solution of an empty set S with weight 0

Recursive calls

- Recursive calls on list …,3, 4, 7



Common Sub-problems

- Only sub-problems are **K(i,w)** for
  - **i = 0,1,..., n**
  - **w = 0,1,..., W**
- Dynamic programming solution
  - Table entry for each **K(i,w)**
    - **OPT** - value of optimal soln for first **i** objects and weight **w**
    - **belong** flag - is x_i a part of this solution?

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  - Table entry for each **K(i,w)**
    - **OPT** - value of optimal soln for first **i** objects and weight **w**
    - **belong** flag - is x_i a part of this solution?
  - Initialize **OPT[0,w]** for **w=0,...,W**
  - Compute all **OPT[i,*]** from **OPT[i-1,*]** for **i>0**

Dynamic Knapsack Algorithm

```
for w=0 to W;  OPT[0,w] ← 0;   end for
for i=1 to n do
    for w=0 to W do
        OPT[i,w]←OPT[i-1,w]
        belong[i,w]←0
        if  w ≥ x_i then
            val ←x_i+OPT[i-1,w-x_i]
            if val>OPT[i,w] then
                OPT[i,w]←val
                belong[i,w]←1
    end for
end for
return(OPT[n,W])
```

Time $O(nW)$

Sample execution on 2, 3, 4, 7 with W=15

Saving Space

- To compute the value OPT of the solution only need to keep the last two rows of OPT at each step

- What about determining the set S?
  - Follow the **belong** flags **O(n)** time
  - What about space?

Three Steps to Dynamic Programming

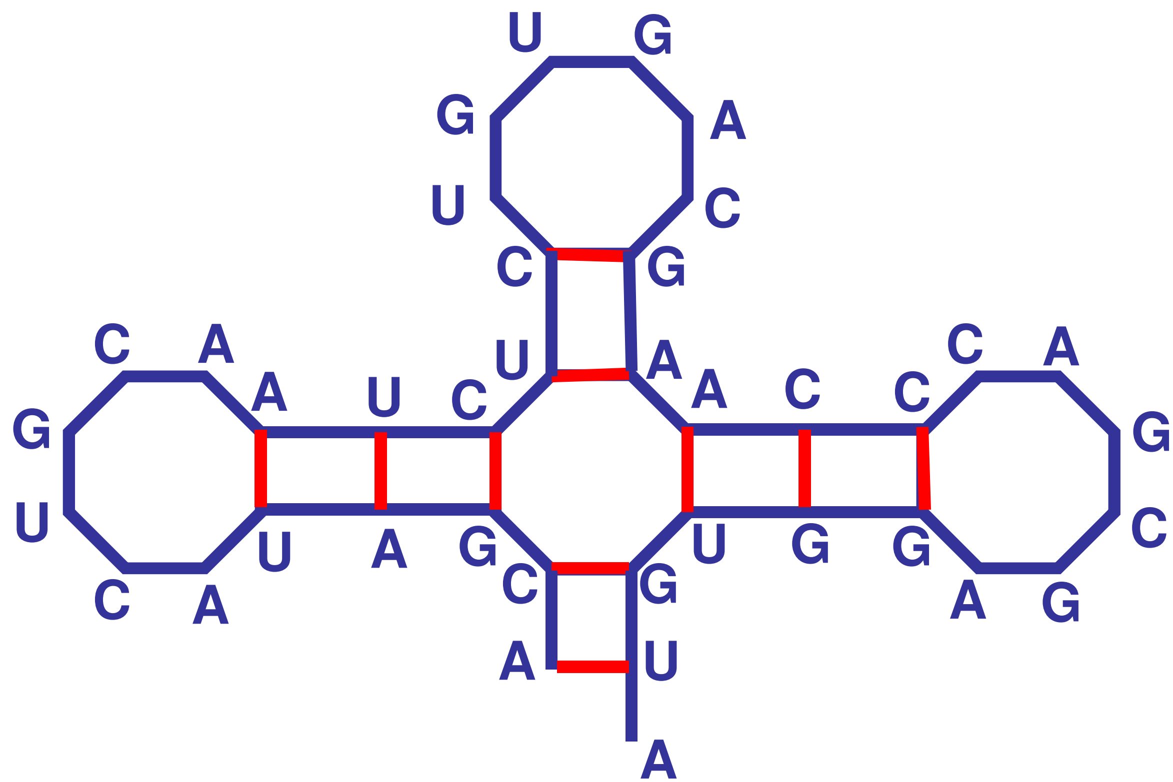
- Formulate the answer as a recurrence relation or recursive algorithm
- Show that the number of different values of parameters in the recursive algorithm is "small"
  - e.g., bounded by a low-degree polynomial
- Specify an order of evaluation for the recurrence so that you already have the partial results ready when you need them.

RNA Secondary Structure: Dynamic Programming on Intervals

- RNA: sequence of bases
  - String over alphabet {**A**, **C**, **G**, **U**}

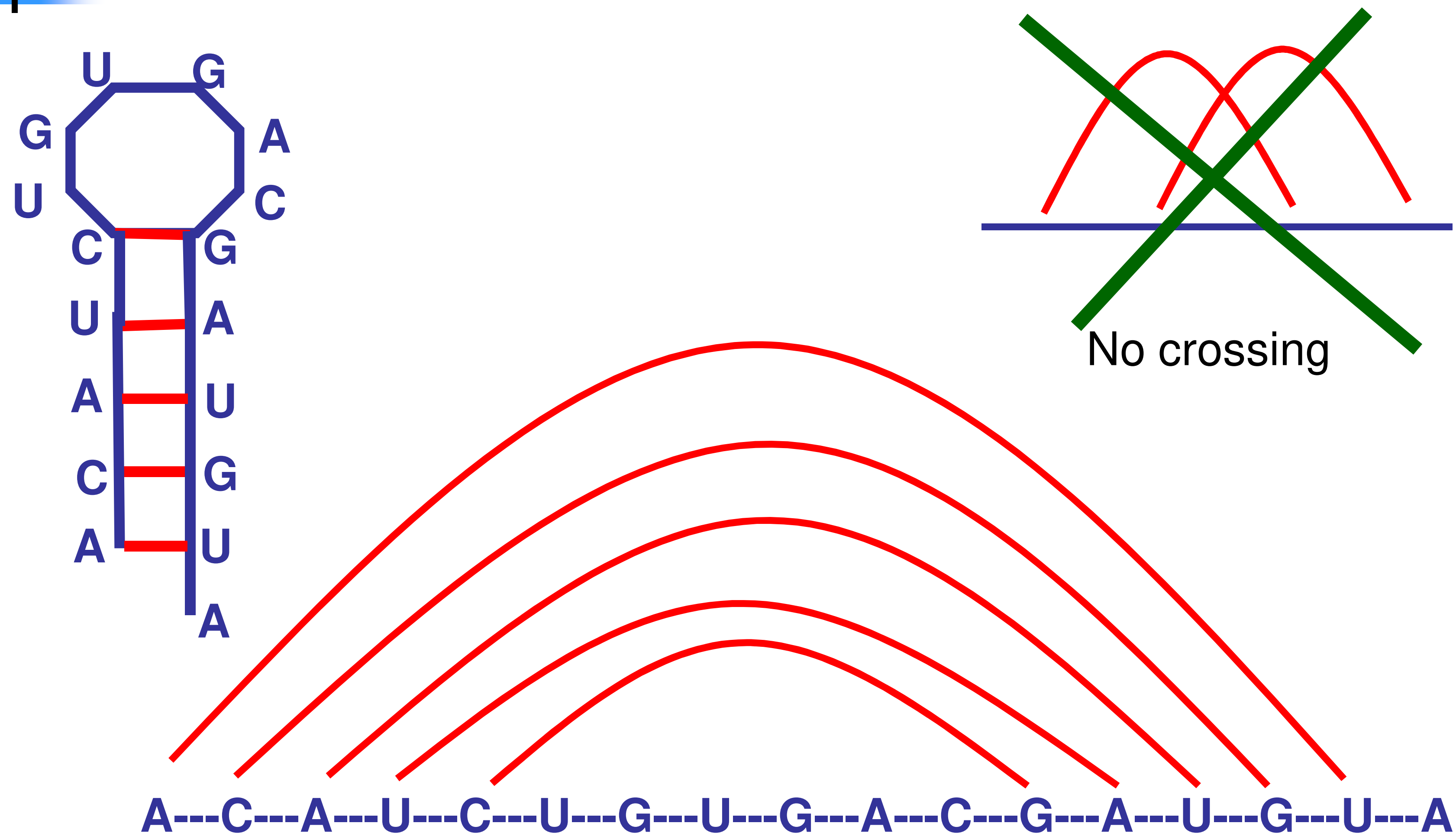
    U-**G**-**U**-**A**-**C**-**C**-**G**-**G**-**U**-**A**-**G**-**U**-**A**-**C**-**A**
- RNA folds and sticks to itself like a zipper
  - **A** bonds to **U**
  - **C** bonds to **G**
  - Bends can’t be sharp
  - No twisting or criss-crossing
- How the bonds line up is called the RNA secondary structure

RNA Secondary Structure



ACGAUACUGCAAUCUCUGUGACGAACCCAGCGAGGUGUA

Another view of RNA Secondary Structure

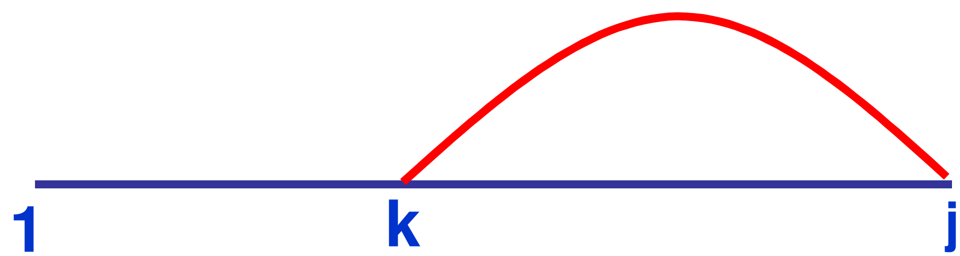


RNA Secondary Structure

- Input: String $x_1 \ldots x_n \in \{A,C,G,U\}^*$
- Output: Maximum size set S of pairs (i,j) such that
  - $\{x_i,x_j\}=\{A,U\}$ or $\{x_i,x_j\} =\{C,G\}$
  - The pairs in S form a matching
  - $i<j-4$ (no sharp bends)
  - No crossing pairs
    - If (i,j) and (k,l) are in S then it is not the case that they cross as in $i<k<j<l$

Recursion Solution

- Try all possible matches for the last bas

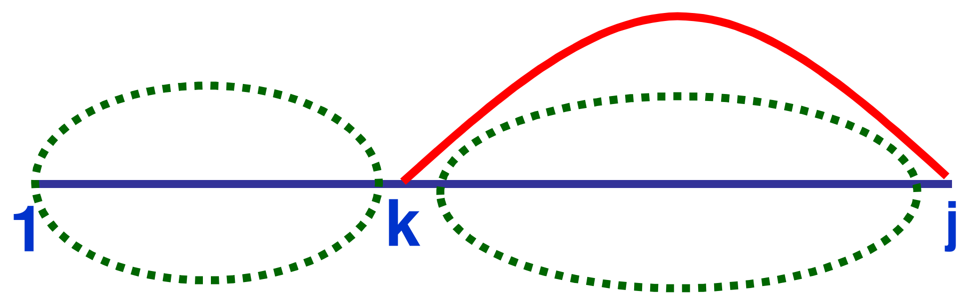


$$OPT(1..j)=MAX(OPT(1..j-1),1+MAX_{k=1..j-5}\,(OPT(1..k-1)+OPT(k+1..j-1))$$

x_k matches x_j

Recursion Solution

- Try all possible matches for the last base



$$\mathrm{OPT}(1..j) = \mathrm{MAX}(\mathrm{OPT}(1..j-1), 1 + \mathrm{MAX}_{k=1..j-5}\ (\mathrm{OPT}(1..k-1) + \mathrm{OPT}(k+1..j-1))$$

x_k matches x_j

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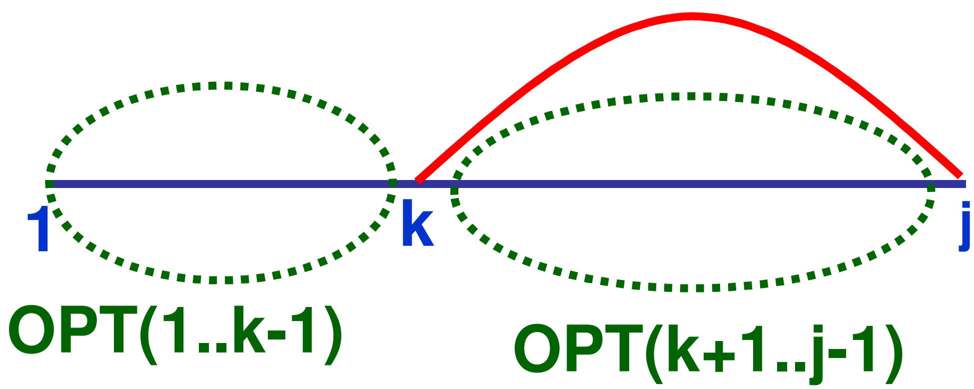

$$OPT(1..j)=MAX\left(OPT(1..j-1),\ 1+\underset{x_k \text{ matches } x_j}{MAX_{k=1..j-5}}\left(OPT(1..k-1)+OPT(k+1..j-1)\right)\right)$$

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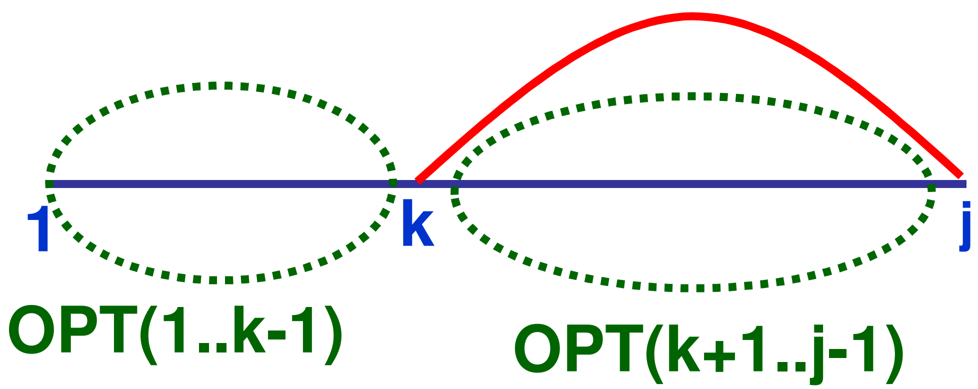


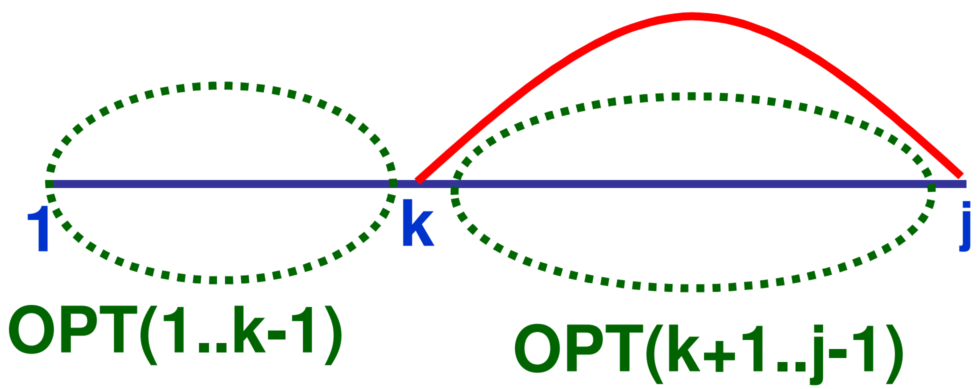
$$OPT(1..j) = MAX(OPT(1..j-1), 1 + MAX_{k=1..j-5}(OPT(1..k-1) + OPT(k+1..j-1)))$$

x_k matches x_j

Doesn't start at 1

Recursion Solution

- Try all possible matches for the last base



$$OPT(1..j) = MAX(OPT(1..j-1), 1 + \underset{x_k \text{ matches } x_j}{MAX_{k=1..j-5}} (OPT(1..k-1) + OPT(k+1..j-1))$$

Doesn't start at 1

General form:

$$OPT(i..j) = MAX(OPT(i..j-1), 1 + \underset{x_k \text{ matches } x_j}{MAX_{k=i..j-5}} (OPT(i..k-1) + OPT(k+1..j-1)))$$

RNA Secondary Structure

- 2D Array **OPT(i,j)** for **i≤j** represents optimal # of matches entirely for segment **i..j**
- For **j-i ≤4** set **OPT(i,j)=0** (no sharp bends)
- Then compute **OPT(i,j)** values when **j-i=5,6,...,n-1** in turn using recurrence.
- Return **OPT(1,n)**
- Total of **O(n³)** time
- Can also record matches along the way to produce **S**
  - Similar polynomial-time algorithm for other problems
    - Context-Free Language recognition
    - Optimal matrix products, etc.
  - All use dynamic programming over intervals

Sequence Alignment: Edit Distance

- Given:
  - Two strings of characters $\mathbf{A=a_1\ a_2\ \ldots\ a_n}$ and $\mathbf{B=b_1\ b_2\ \ldots\ b_m}$
- Find:
  - The minimum number of edit steps needed to transform **A** into **B** where an edit can be:
  - insert a single character
  - delete a single character
  - substitute one character by another

Applications

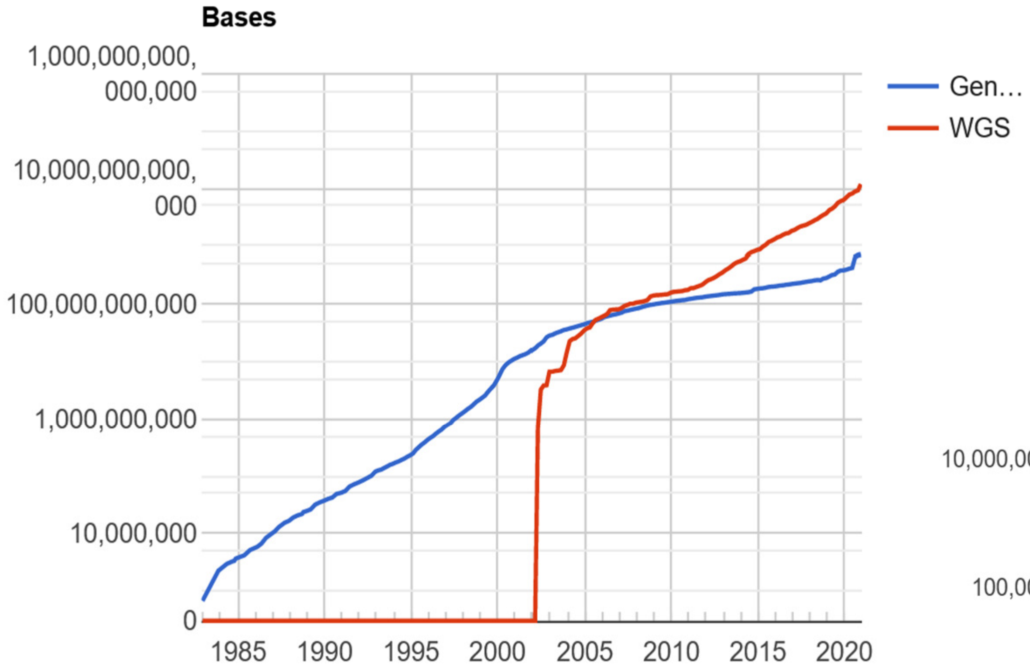
- "diff" utility – where do two files differ
- Version control & patch distribution – save/send only changes
- Molecular biology
  - Similar sequences often have similar origin and function
  - Similarity often recognizable despite millions or billions of years of evolutionary divergence

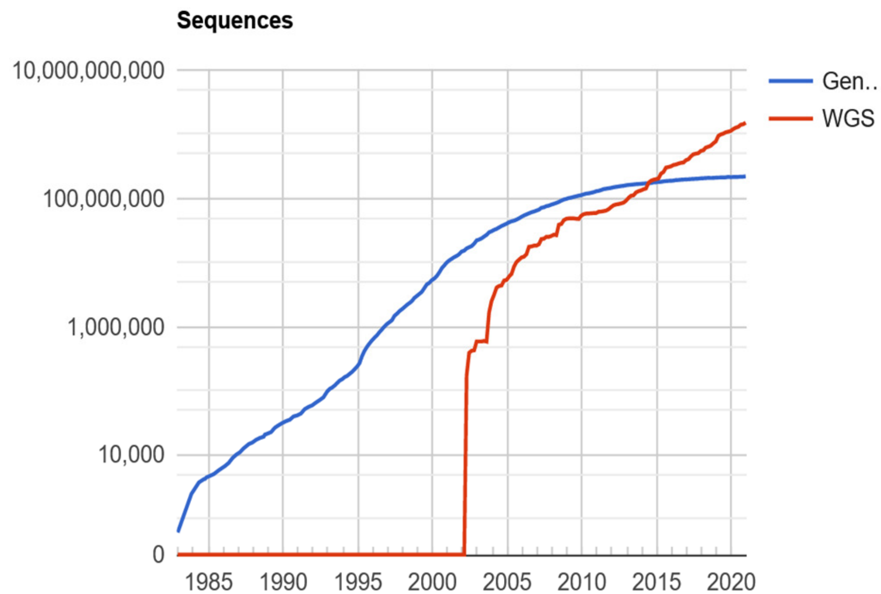
```
C A - C G T G A T
| |   | |   | X |
C A T C G - G T T
```

Sequence Alignment vs Edit Distance

- Sequence Alignment
  - Insert corresponds to aligning with a "**–**" in the first string
    - Cost δ (in our case **1**)
  - Delete corresponds to aligning with a "**–**" in the second string
    - Cost δ (in our case **1**)
  - Replacement of an **a** by a **b** corresponds to a mismatch
    - Cost α_{ab} (in our case **1** if $a \neq b$ and **0** if $a = b$)
- In Computational Biology this alignment algorithm is attributed to Smith & Waterman

GenBank and WGS Statistics





Recursive Solution

- Sub-problems: Edit distance problems for **all prefixes** of **A** and **B** that don't include all of both **A** and **B**

- Let **D(i,j)** be the number of edits required to transform $a_1\ a_2\ \ldots\ a_i$ into $b_1\ b_2\ \ldots\ b_j$

- Clearly **D(0,0)=0**

Computing D(n,m)

- Imagine how best sequence handles the last characters a_n and b_m
- If best sequence of operations
  - deletes a_n then $D(n,m)=D(n-1,m)+1$
  - inserts b_m then $D(n,m)=D(n,m-1)+1$
  - replaces a_n by b_m then
    $D(n,m)=D(n-1,m-1)+1$
  - matches a_n and b_m then
    $D(n,m)=D(n-1,m-1)$

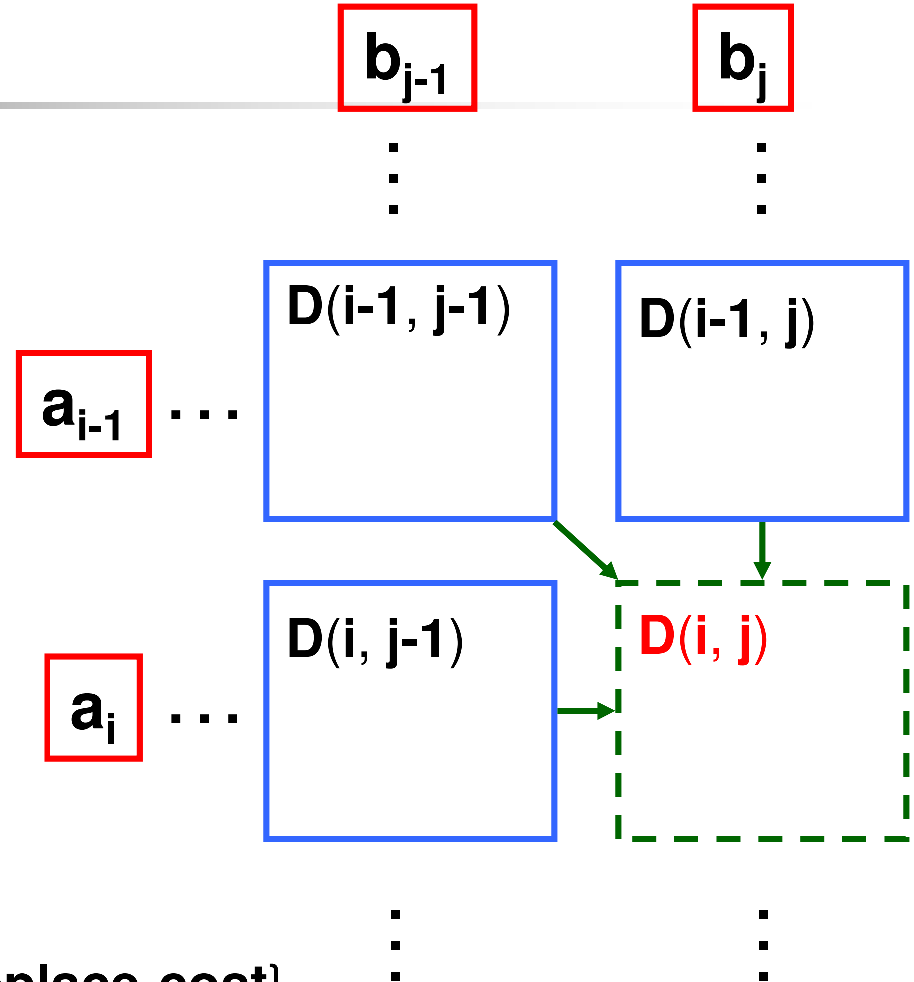
Recursive algorithm D(n,m)

```
if  n=0  then
    return (m)
elseif  m=0  then
    return(n)
else
    if  a_n=b_m  then
        replace-cost ← 0
    else
        replace-cost ← 1
    endif
    return(min{ D(n-1, m) + 1,
                D(n, m-1) +1,
                D(n-1, m-1) + replace-cost})
```

$\left.\begin{array}{l} \textbf{replace-cost} \leftarrow 0 \\ \text{else} \\ \textbf{replace-cost} \leftarrow 1 \end{array}\right\}$ cost of substitution of $\mathbf{a}_n$ by $\mathbf{b}_m$ (if used)

Dynamic Programming

```
for j = 0 to m;  D(0,j) ← j; endfor
for i = 1 to n;  D(i,0) ← i; endfor
for i = 1 to n
    for j = 1 to m
        if  a_i=b_j then
            replace-cost ← 0
        else
            replace-cost ← 1
        endif
        D(i,j) ←  min { D(i-1, j) + 1,
                        D(i, j-1) + 1,
                        D(i-1, j-1) + replace-cost}
    endfor
endfor
```



Example run with AGACATTG and GAGTTA

			A	G	A	C	A	T	T	G
		0	1	2	3	4	5	6	7	8
	0									
G	1									
A	2									
G	3									
T	4									
T	5									
A	6									

Example run with AGACATTG and GAGTTA

		A	G	A	C	A	T	T	G
	0	1	2	3	4	5	6	7	8
G	1	1	1	2	3	4	5	6	7
A	2								
G	3								
T	4								
T	5								
A	6								

Example run with AGACATTG and GAGTTA

		A	G	A	C	A	T	T	G
	0	1	2	3	4	5	6	7	8
G	1	1	1	2	3	4	5	6	7
A	2	1	2	1					
G	3								
T	4								
T	5								
A	6								

Example run with AGACATTG and GAGTTA

		A	G	A	C	A	T	T	G
	0	1	2	3	4	5	6	7	8
G	1	1	1	2	3	4	5	6	7
A	2	1	2	1	2	3	4	5	6
G	3	2	1	2	2	3	4	5	5
T	4								
T	5								
A	6								

Example run with AGACATTG and GAGTTA

		A	G	A	C	A	T	T	G
	0	1	2	3	4	5	6	7	8
G	1	1	1	2	3	4	5	6	7
A	2	1	2	1	2	3	4	5	6
G	3	2	1	2	2	3	4	5	5
T	4	3	2	2	3	3	3	4	5
T	5	4	3	3	3	4	3	3	4
A	6	5	4	3	4	3	4	4	4

Example run with AGACATTG and GAGTTA

		A	G	A	C	A	T	T	G
	0	1	2	3	4	5	6	7	8
G	1	1	1	2	3	4	5	6	7
A	2	1	2	1	2	3	4	5	6
G	3	2	1	2	2	3	4	5	5
T	4	3	2	2	3	3	3	4	5
T	5	4	3	3	3	4	3	3	4
A	6	5	4	3	4	3	4	4	4

Example run with AGACATTG and GAGTTA

		A	G	A	C	A	T	T	G
	0	**1**	2	3	4	5	6	7	8
G	1	1	**1**	2	3	4	5	6	7
A	2	1	2	**1**	2	3	4	5	6
G	3	2	1	2	**2**	**3**	4	5	5
T	4	3	2	2	3	3	**3**	4	5
T	5	4	3	3	3	4	3	**3**	4
A	6	5	4	3	4	3	4	4	**4**

Reading off the operations

- Follow the sequence and use each color of arrow to tell you what operation was performed.
- From the operations can derive an optimal alignment

A G A C A T T G
_ G A G _ T T A

Saving Space

- To compute the distance values we only need the last two rows (or columns)
  - **O**(min(**m**,**n**)) space
- To compute the alignment/sequence of operations
  - seem to need to store all **O**(**mn**) pointers/arrow colors
- Nifty divide and conquer variant that allows one to do this in **O**(min(**m**,**n**)) space and retain **O**(**mn**) time
  - In practice the algorithm is usually run on smaller chunks of a large string, e.g. **m** and **n** are lengths of genes so a few thousand characters
    - Researchers want all alignments that are close to optimal
    - Basic algorithm is run since the whole table of pointers (**2** bits each) will fit in RAM
  - Ideas are neat, though

Saving space

- Alignment corresponds to a path through the table from lower right to upper left
  - Must pass through the middle column
- Recursively compute the entries for the middle column from the left
  - If we knew the cost of completing each then we could figure out where the path crossed
  - Problem
    - There are **n** possible strings to start from.
  - Solution
    - Recursively calculate the right half costs for each entry in this column using alignments starting at the other ends of the two input strings!
  - Can reuse the storage on the left when solving the right hand problem

Shortest paths with negative cost edges (Bellman-Ford)

- Dijsktra's algorithm failed with negative-cost edges
  - What can we do in this case?
  - Negative-cost cycles could result in shortest paths with length **-∞**
- Suppose no negative-cost cycles in G
  - Shortest path from **s** to **t** has at most **n-1** edges
    - If not, there would be a repeated vertex which would create a cycle that could be removed since cycle can't have –ve cost

Shortest paths with negative cost edges (Bellman-Ford)

- We want to grow paths from **s** to **t** based on the # of edges in the path
- Let Cost(**s**,**t**,**i**)=cost of minimum-length path from **s** to **t** using up to **i** hops.
  - $\text{Cost}(\mathbf{v},\mathbf{t},\mathbf{0}) = \begin{cases} \mathbf{0} \text{ if } \mathbf{v}=\mathbf{t} \\ \infty \text{ otherwise} \end{cases}$
  - $\text{Cost}(\mathbf{v},\mathbf{t},\mathbf{i}) = \min\{\text{Cost}(\mathbf{v},\mathbf{t},\mathbf{i-1}), \min_{(\mathbf{v},\mathbf{w})\in \mathbf{E}}(\mathbf{c}_{\mathbf{vw}} + \text{Cost}(\mathbf{w},\mathbf{t},\mathbf{i-1}))\}$

Bellman-Ford

- Observe that the recursion for Cost(**s**,**t**,**i**) doesn't change **t**
  - Only store an entry for each **v** and **i**
    - Termed OPT(**v**,**i**) in the text
- Also observe that to compute OPT(*,i) we only need OPT(*,i-1)
  - Can store a current and previous copy in O(**n**) space.

Bellman-Ford

ShortestPath(G,s,t)

for all $\mathbf{v} \in \mathbf{V}$

$\mathbf{OPT}[\mathbf{v}] \leftarrow \infty$

$\mathbf{OPT}[\mathbf{t}] \leftarrow \mathbf{0}$

for $\mathbf{i}=\mathbf{1}$ to $\mathbf{n-1}$ do

for all $\mathbf{v} \in \mathbf{V}$ do

$\mathbf{OPT'}[\mathbf{v}] \leftarrow \min_{(\mathbf{v},\mathbf{w}) \in \mathbf{E}} (\mathbf{c}_{\mathbf{vw}} + \mathbf{OPT}[\mathbf{w}])$

for all $\mathbf{v} \in \mathbf{V}$ do

$\mathbf{OPT}[\mathbf{v}] \leftarrow \min(\mathbf{OPT'}[\mathbf{v}], \mathbf{OPT}[\mathbf{v}])$

return $\mathbf{OPT}[\mathbf{s}]$

$O(mn)$ time

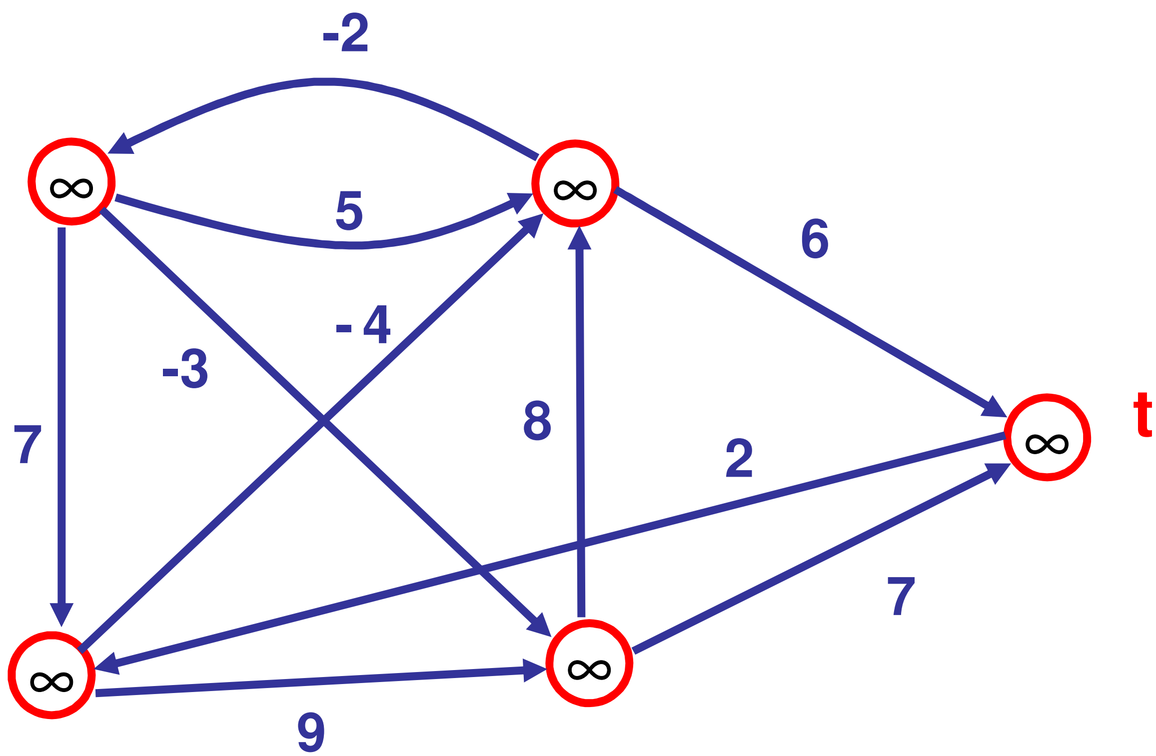
Negative cycles

- Claim: There is a negative-cost cycle that can reach t iff for some vertex $v \in V$, $Cost(v,t,n) < Cost(v,t,n-1)$
- Proof:
  - We already know that if there aren't any then we only need paths of length up to **n-1**
  - For the other direction
    - The recurrence computes **Cost(v,t,i)** correctly for **any** number of hops **i**
    - The recurrence reaches a fixed point if for every $v \in V$, $Cost(v,t,i) = Cost(v,t,i-1)$
    - A negative-cost cycle means that eventually some **Cost(v,t,i)** gets smaller than any given bound
      - Can't have a –ve cost cycle if for every $v \in V$, $Cost(v,t,n) = Cost(v,t,n-1)$

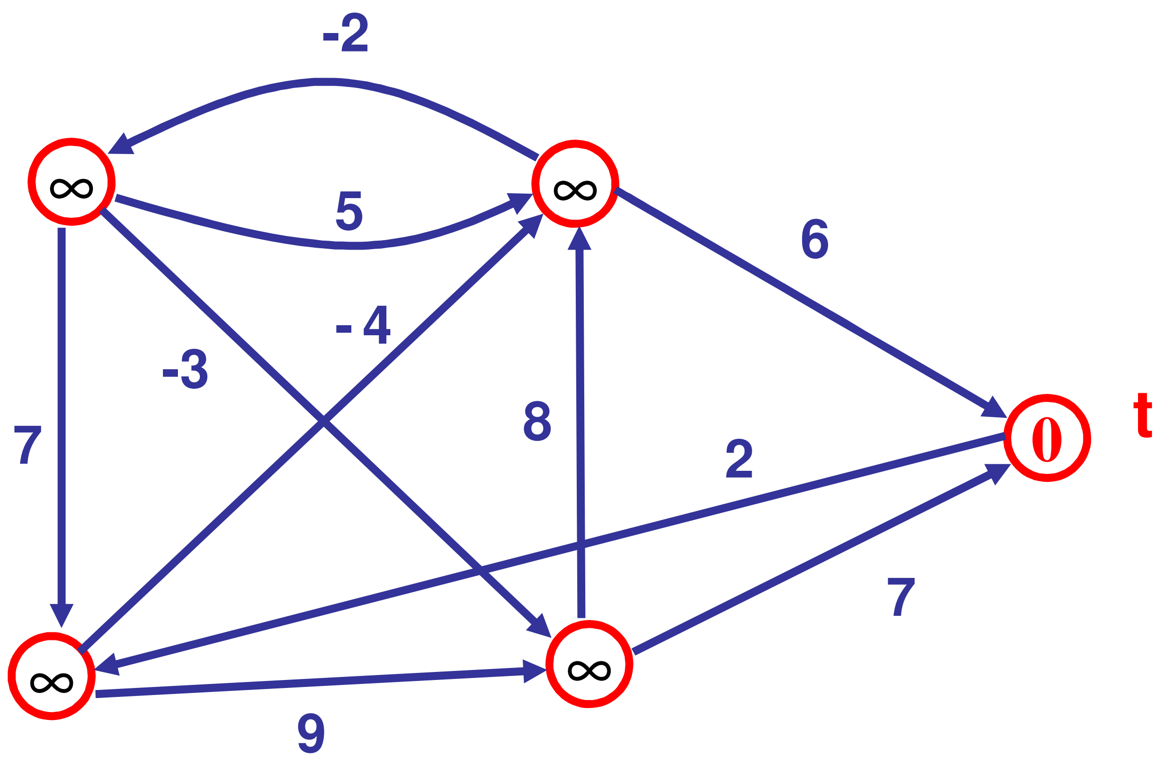
Last details

- Can run algorithm and stop early if the OPT and OPT' arrays are ever equal
  - Even better, one can update only neighbors **v** of vertices **w** with OPT'[**w**]≠OPT[**w**]
- Can store a successor pointer when we compute OPT
  - Homework assignment
- By running for step **n** we can find some vertex **v** on a negative cycle and use the successor pointers to find the cycle

Bellman-Ford



Bellman-Ford



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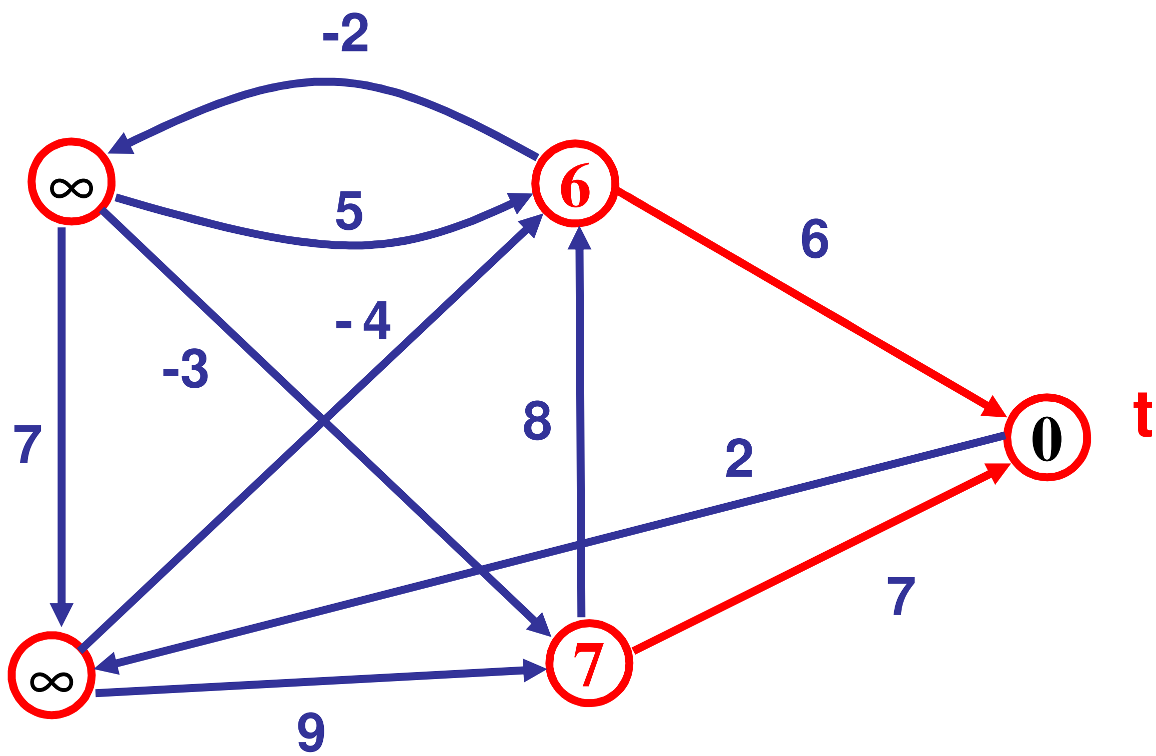

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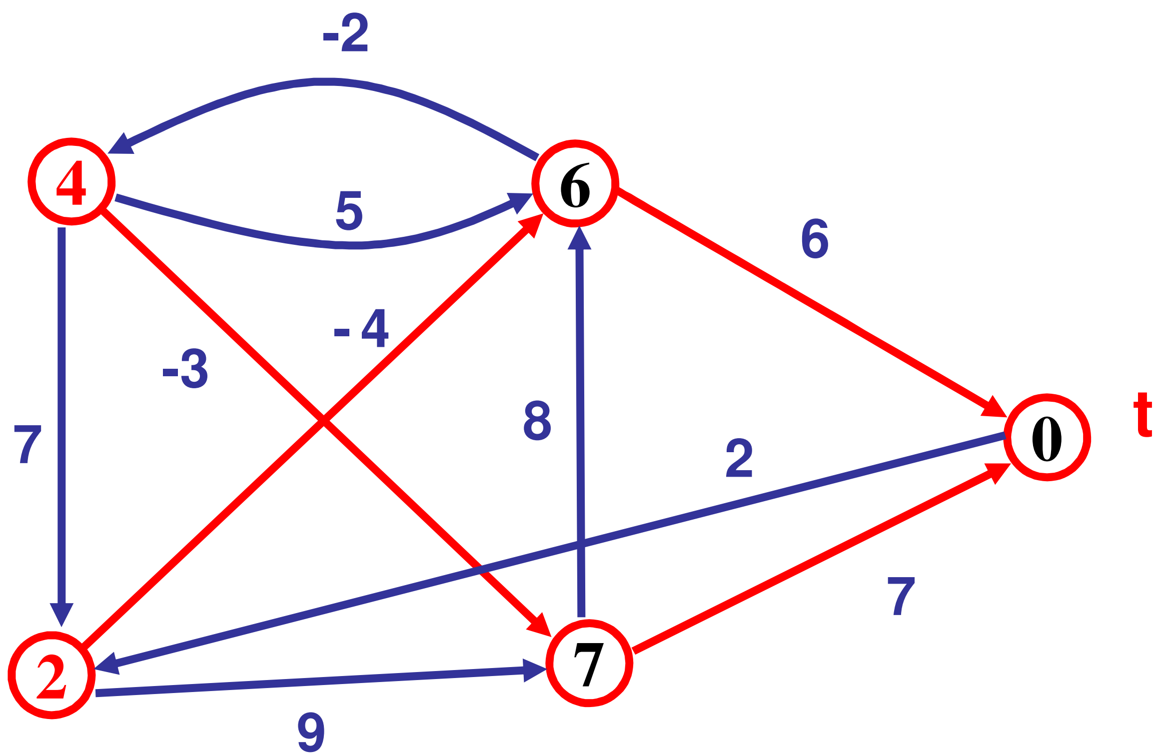

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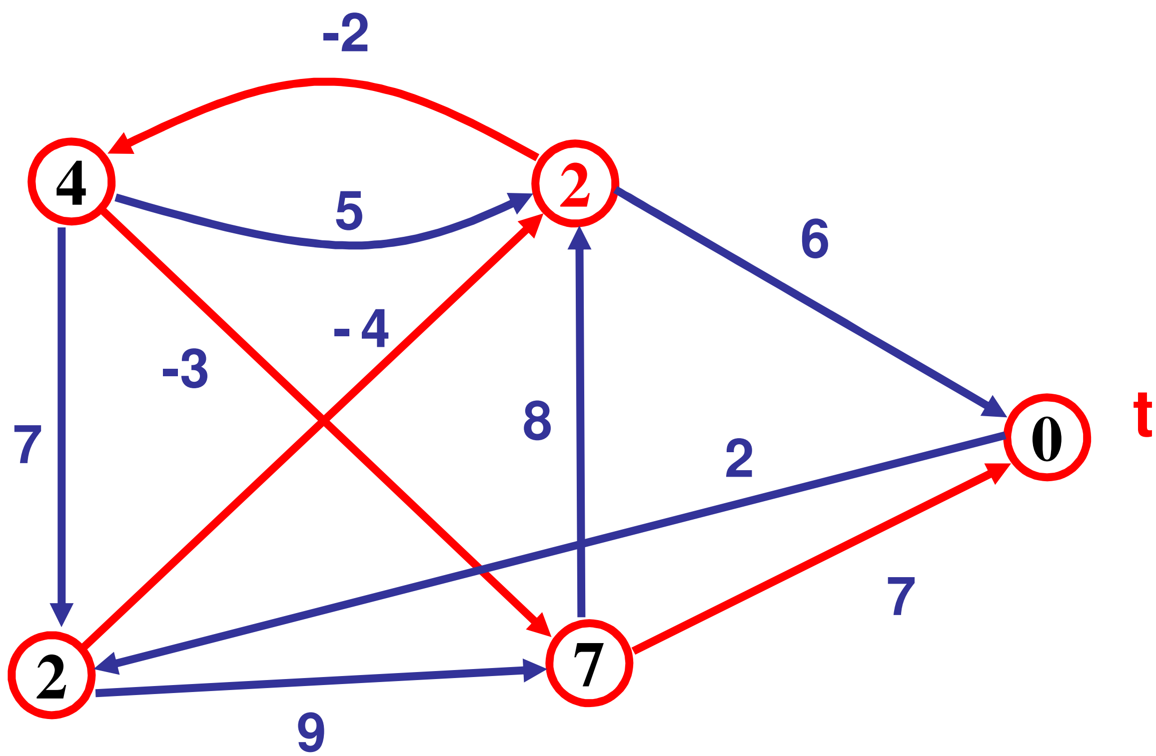

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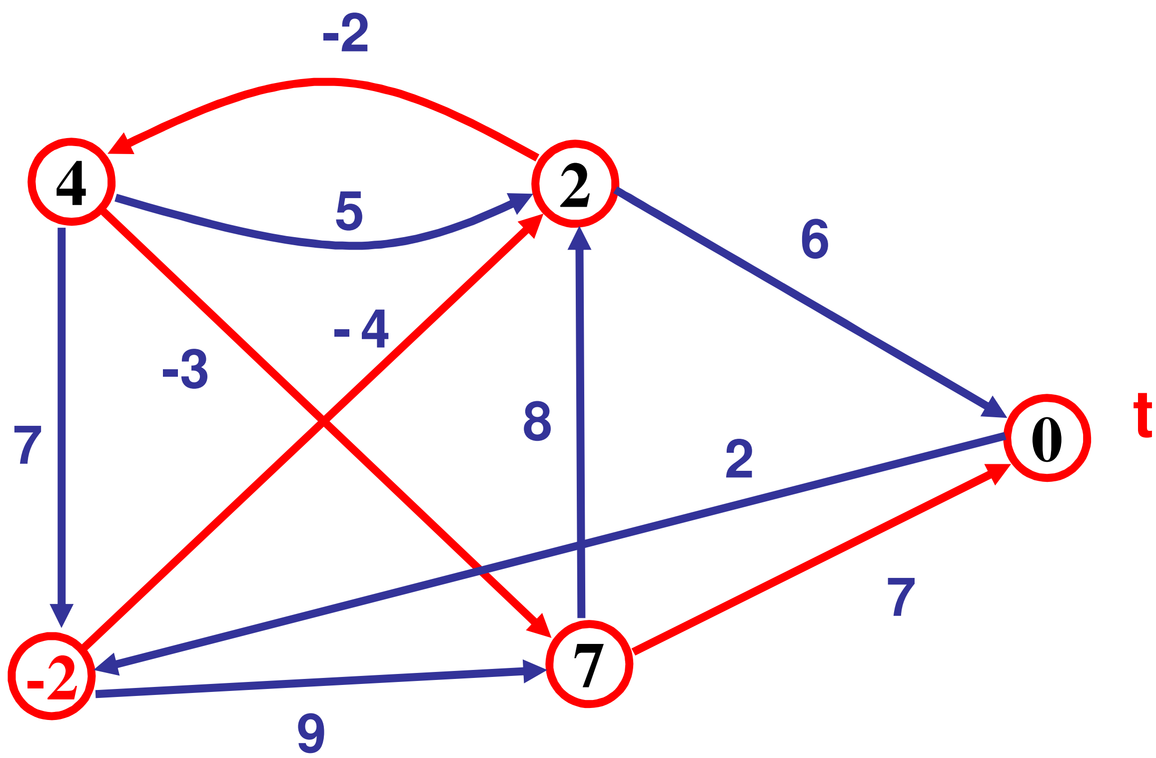

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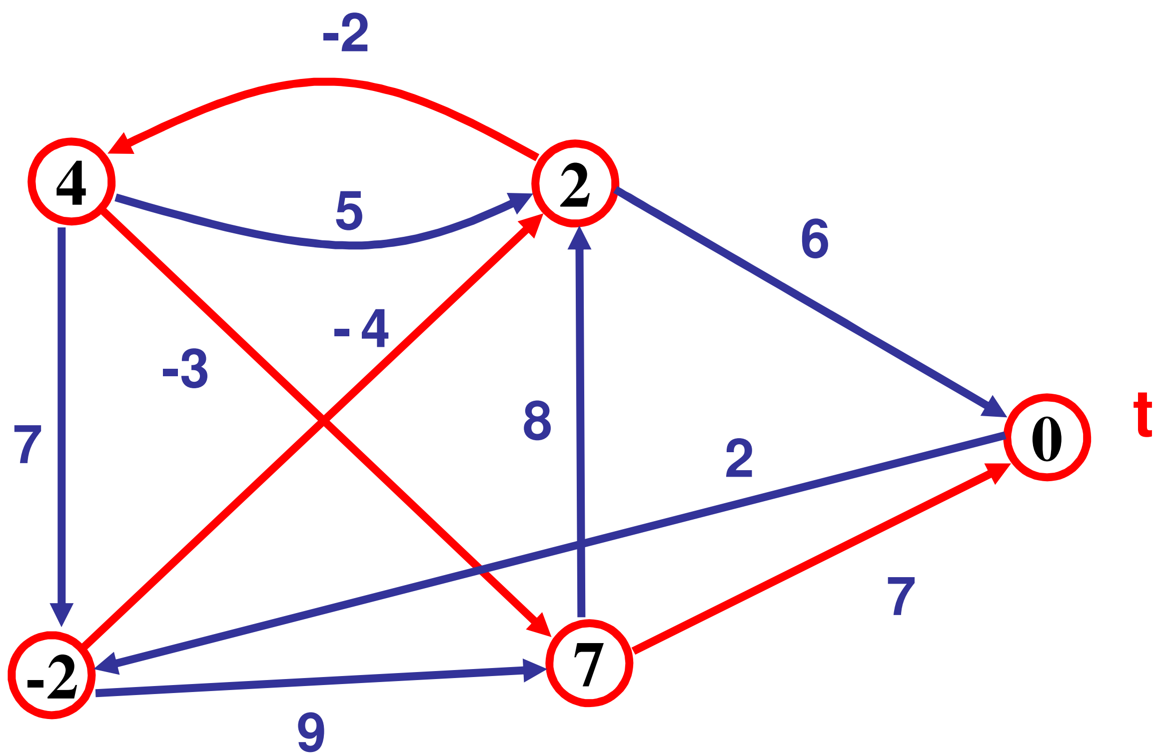

Bellman-Ford with a DAG

Edges only go from lower to higher-numbered vertices

- Update distances in reverse order of topological sort
- Only one pass through vertices required
- $O(\mathbf{n}+\mathbf{m})$ time