CSE421: Design and Analysis of Algorithms	April 19, 2021
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1 Interval Scheduling

Theorem 1. *Greedy is optimum.*

Proof Technique: Greedy stays ahead, i.e., Greedy is "better" than OPT at any time in ALG. **Proof** Let $i_1, \ldots, i_k$ be the jobs chosen by the greedy algorithm. Similarly, let $j_1, \ldots, j_m$ be the jobs chosen by OPT. Our goal is to show $k \geq m$. Recall that by definition of OPT we always know that $m \geq k$. So, we if we prove $k \geq m$ these together imply that $k = m$ and we are done.

Before proving $m \geq k$ we prove the following claim by induction:

Claim 2. *For any $1 \leq r \leq k$, we have $f(i_r) \leq f(j_r)$.*

Proof Let $P(r) := f(i_r) \leq f(j_r)$.

Base Case: P(1) holds. This is because the first job in Greedy is the job with the smallest finishing time.

IH: Assume $P(r)$ for some $r \geq 1$.

IS: Our goal is to prove $P(r+1)$, i.e., to show $f(i_{r+1}) \leq f(j_{r+1})$. First, by IH we can write,

$$f(i_r) \leq f(j_r) \leq s(j_{r+1}), \tag{1}$$

where the second equation follows by the fact that the jobs scheduled in OPT are non-overlapping.

Now, by definition of Greedy, i_{r+1} is the job with the smallest finishing time among all jobs that start right at or after $f(i_r)$. Equation 1 shows that the job j_{r+1} also starts right at or after $f(i_r)$; so j_{r+1} is a candidate for i_{r+1} and so we must have $f(i_{r+1}) \leq f(j_{r+1})$. ■

Now, we are ready to prove $k \geq m$.

For contradiction assume that $k < m$. So, OPT has a job j_{k+1} but i_{k+1} does not exists. By the above claim we have $f(i_k) \leq f(j_k) \leq s(j_{k+1})$, i.e., there is a job in OPT which starts right at or after $f(i_k)$. So, greedy could have scheduled j_{k+1} after i_k but it didn't which is a contradiction. So, we must have $k \geq m$. ■