CSE 421: Introduction to Algorithms

Stable Matching

Shayan Oveis Gharan

Administrativia Stuffs

Lectures: M/W/F 1:30-2:20
Zoom Id: https://washington.zoom.us/j/9771906541

Office hours: M/W 2:30-3:20, T 4:30-5:20
https://washington.zoom.us/j/94137597308

Discussion Board: Use edstem https://edstem.org


CSE 421: Introduction to Algorithms

Winter, 2018

Shayan Oveis Gharan

MWF 2:30-3:20, MGH 389
Office hours in CSE 636
M/W/F 3:30-4:20

Textbook:

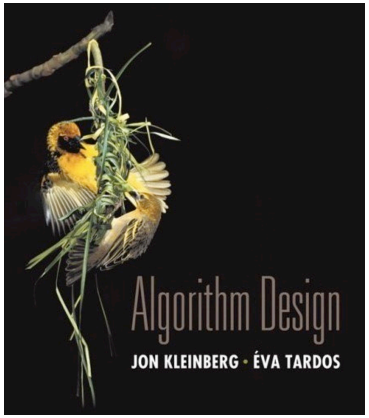
Algorithm Design by Jon Kleinberg and Eva Tardos, Addison-Wesley, 2006. We will cover almost all of chapters 1-8 of the Kleinberg/Tardos text plus some additional material from later chapters. In addition, I recommend reading chapter 5 of Introduction to Algorithms: A Creative Approach, by Udi Manber, Addison-Wesley 1989. This book has a unique point of view on algorithm design.

Another handy reference is Steven Skiena's Stonybrook Algorithm Repository

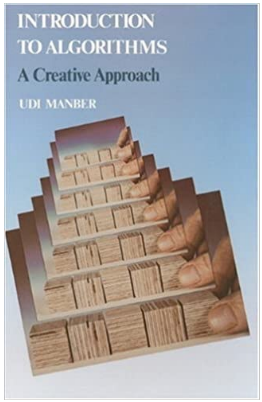
Grading Scheme (Roughly):

Homework 50%
Midterm 15-20%
Final Exam 30-35%

cs.washington.edu/421



Course textbook



Supplementary text

TAs

Mrigank Arora	Thu 10:00-10:50
Todor Dimitrov	Tue 9:00-9:50
Aidan Gottlieb	Tue 2:30-3:20
Johnson Kuang	Thu 11:00-11:50
Chase Lee	Wed 8:00-8:50
Mickey Moonkaen	Mon 4:00-4:50
Yunkyu Song	Tue 5:30-6:20
Savanna Yee	Mon 10:00-10:50
Liangyu Zhao	Wed 4:00-4:50
Albert Zhong	Tues 12:30-1:20

Grading

- Weekly HWs, First HW due April 8th
- Submit to Gradescope
- Midterm (05/04/2020), Final (06/08/2020)
  - Exams are open book, open note, no internet access
  - Midterm 50+15 minutes, Final 110+15 minutes.
  - Will have two exams at two time-zones: usual and 10:00 PM PST
- HW 50%, Midterm 15-20%, Final 30-35%
- Extra Credit problems can boost your final GPA by 0.1

Daily Quizzes

- One quiz before every lecture
- 1-2 questions about the materials of the previous lecture
- Typically yes/no or multiple choice

- Login to canvas (assignment tab) to access the quiz

- Available 1:25-1:30 (before lecture), you have 3-4 minutes to answer

- Daily Quizes can boost up your final GPA by 0.1
- If you don’t answer any of them you can still get 4.0!

Practicing with Zoom!

- Everyone is muted by default!
- Please share your video!
- Please ask your questions (not in chat)
- Videos: Recorded and can be access in Canvas (zoom tab)
- Zoom Breakouts: Small groups to work on in-class exercises

Stable Matching Problem

Stable Matching Problem

Given n companies $c_1, \dots, c_n$,
and n applicants, $a_1, \dots, a_n$
find a "stable matching".

- Participants rate members of opposite group.
- Each company lists applicants in order of preference.
- Each applicant lists companies in order of preference.

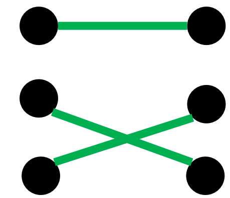
	favorite		least favorite
	1st	2nd	3rd
c_1	a_1	a_2	a_3
c_2	a_2	a_1	a_3
c_3	a_1	a_2	a_3

	favorite		least favorite
	1st	2nd	3rd
a_1	c_2	c_1	c_3
a_2	c_1	c_2	c_3
a_3	c_1	c_2	c_3

Stable Matching

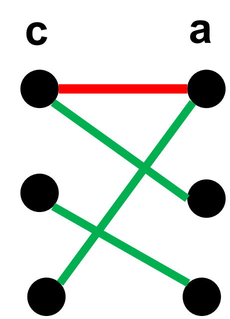
Perfect matching:

- Each company gets exactly one applicant.
- Each applicant gets exactly one company.



Stability: no incentive for some pair of participants to undermine assignment by joint action.

In a matching M, an unmatched pair a-c is unstable if a and c prefer each other to current partners.



Stable matching: perfect matching with no unstable pairs.

Stable matching problem: Given the preference lists of n companies and n applicants, find a stable matching if one exists.

Example

Question. Is assignment $(c_1, a_3), (c_2, a_2), (c_3, a_1)$ stable?

favorite ↓ … least favorite ↓

	1st	2nd	3rd
c_1	a_1	a_2	a_3
c_2	a_2	a_1	a_3
c_3	a_1	a_2	a_3

favorite ↓ … least favorite ↓

	1st	2nd	3rd
a_1	c_2	c_1	c_3
a_2	c_1	c_2	c_3
a_3	c_1	c_2	c_3

Example

Question. Is assignment $(c_1, a_3), (c_2,a_2), (c_3,a_1)$ stable?

Answer. No. a_2, c_1 will hook up.

favorite ↓ … least favorite ↓

	1st	2nd	3rd
c_1	a_1	a_2	a_3
c_2	a_2	a_1	a_3
c_3	a_1	a_2	a_3

favorite ↓ … least favorite ↓

	1st	2nd	3rd
a_1	c_2	c_1	c_3
a_2	c_1	c_2	c_3
a_3	c_1	c_2	c_3

Example

Question: Is assignment $(c_1, a_1), (c_2,a_2), (c_3,a_3)$ stable?

Answer: Yes.

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
c_1	a_1	a_2	a_3
c_2	a_2	a_1	a_3
c_3	a_1	a_2	a_3

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
a_1	c_2	c_1	c_3
a_2	c_1	c_2	c_3
a_3	c_1	c_2	c_3

Existence of Stable Matchings

Question. Do stable matchings always exist?
Answer. Yes, but not obvious a priori.

Stable roommate problem:

2n people; each person ranks others from **1** to **2n-1**.
Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
David	A	B	C

A-B, C-D ⇒ B-C unstable
A-C, B-D ⇒ A-B unstable
A-D, B-C ⇒ A-C unstable

So, Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm [Gale-Shapley'62]

```
Initialize each side to be free.
while (some company is free and hasn't proposed to every
applicant) {
    Choose such a c
    a = 1st woman on c's list to whom c has not yet proposed
    if (a is free)
        assign c and a
    else if (a prefers c to her current c')
        assign c and a, and c' to be free
    else
        a rejects c
}
```

First step: Properties of Algorithm

Observation 1: Companies propose to Applicants in decreasing order of preference.

Observation 2: Each company proposes to each applicant at most once

Observation 3: Once an applicant is matched, she never becomes unmatched; she only "trades up."

What do we need to prove?

1) The algorithm ends
   - How many steps does it take?

2) The algorithm is correct [usually the harder part]
   - It outputs a perfect matching
   - The output matching is stable

1) Termination

Claim. Algorithm terminates after $\leq \boldsymbol{n^2}$ iterations of while loop.

Proof. Observation 2: Each company proposes to each applicant at most once.

Each company makes at most n proposals

So, there are only n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
Vmware	A	B	C	D	E
Walmart	B	C	D	A	E
Xfinity	C	D	A	B	E
Yamaha	D	A	B	C	E
Zoom	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Brenda	X	Y	Z	V	W
Claire	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

2) Correctness: Output is Perfect matching

Claim. All Companies and Applicants get matched.

Proof. (by contradiction)

Suppose, for sake of contradiction, that c_1 is not matched upon termination of algorithm.

Then some applicant, say a_1, is not matched upon termination.

By Observation 3 (only trading up, never becoming unmatched), a_1 was never proposed to.

But, c_1 proposes to everyone, since it ends up unmatched.

■

2) Correctness: Stability

Claim. No unstable pairs.

Proof. (by contradiction)

Suppose c, a is an unstable pair: each prefers each other to the partner in Gale-Shapley matching **S***.

Obs1: companies propose in decreasing order of preference

Case 1: c never proposed to a.

$\Rightarrow$ c prefers its **S*** partner to a.

$\Rightarrow$ c, a is stable.

Case 2: c proposed to a.

$\Rightarrow$ a rejected c (right away or later)

$\Rightarrow$ a prefers her **S*** partner to c.

$\Rightarrow$ c, a is stable.

Obs3: applicants only trade up

In either case c, a is stable, a contradiction. ■

Summary

Stable matching problem: Given n companies and n applicants, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm: Guarantees to find a stable matching for any problem instance.

- Q: How to implement GS algorithm efficiently?
- Q: If there are multiple stable matchings, which one does GS find?
- Q: How many stable matchings are there?

Matching Residents to Hospitals

Goal: Given a set of preferences among hospitals and medical school residents (graduating medical students), design a self-reinforcing admissions process.

Unstable pair: applicant **A** and hospital **Y** are unstable if:

- **A** prefers **Y** to its assigned hospital.
- **Y** prefers **A** to one of its admitted applicants.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital side deal from being made.