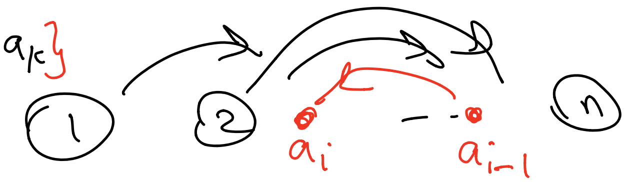
Proof Techniques
- Direct Proof
- Proof by Contradi: When there are many bad events that we don't want to happen.
- Induction: when $\exists$ way to reduce instance of size n to $n-1$.
- Reduction (we will see later)

Claim: If G has a topological order $\Rightarrow$ G is a DAG.

Pf: By contradiction (BC many possible bad events).

Supp G has topol order

Assume G has a cycle: $\{a_1, a_2 - a_k\}$

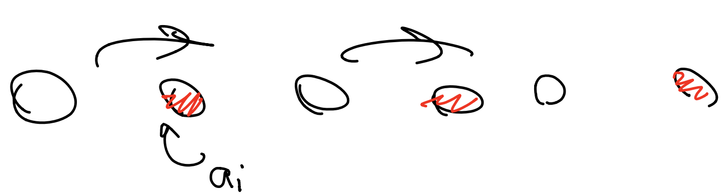


Supp $a_i = \min\{a_1 \text{--}, a_k\}$

Then (a_{i-1}, a_i) is an edge. But $a_{i-1} > a_i$

So this is left edge contradiction!

a_i shows up first in topolog order

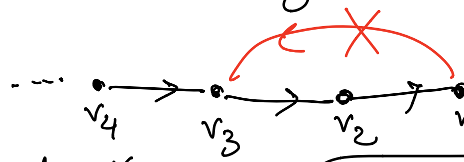


Lem: G is a DAG $\Rightarrow$ G has a node with indeg $= 0$.

Pf: (by contradi BC n possible bad events when node i has indeg > 0)

Supp G is a DAG, & every node i has indeg > 0.

Start with v_1



Similar Pf.

If G (undirected) no cycle $\Rightarrow \exists v : \deg(v) \leq 1$.

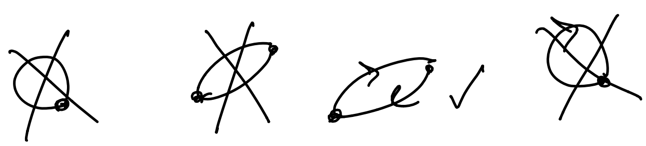
$\text{Indeg}(v_1) > 0 \Rightarrow \exists v_2$ s.t. $v_2 \to v_1$

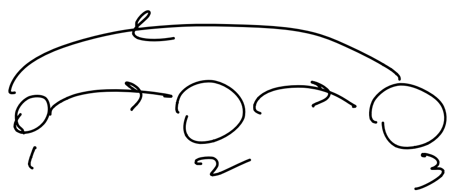
$\text{Indeg}(v_2) > 0$ → $v_1 \to v_2$ makes a cycle X
→ $\exists v_3 : v_3 \to v_2$

$\text{Indeg}(v_3) > 0$ → $v_1 \to v_3$, $v_2 \to v_3$ makes a cycle X
→ $\exists v_4 : v_4 \to v_3$.

⋮

G has finite many vertices so either we get to a node of indeg $= 0$ or a cycle (contradiction!) □





Lem: G is a DAG $\Rightarrow$ G has a topological order.

Pf: By Induction

$P(n)$ = "Any DAG with n nodes have a topological order".

Base Case $P(1)$: • ✓

IH: $P(n-1)$.

IS: Goal $P(n)$. Let G be an arbitrary DAG.

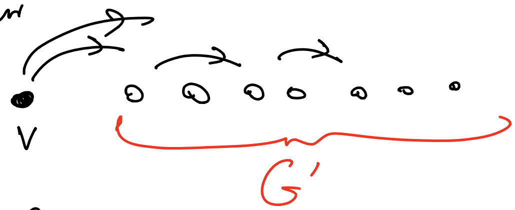
By prev Lemm, G has a source node say v.

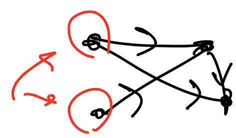
Let $G' = G - v$. → G' has $n-1$ nodes

→ G' is acyclic: BC by removing edges/verti

So by IH: G′ has ~~a~~ topo. order   we don't create cycle.

put v at the beginning
this is a valid topo. order
of G ~~see~~ since indeg(v) = 0



Source 

DAG ≈ Trees

Source   Leaves

Indn by rem source   Indn by remo. ln