1 In-class Exercise

1. Recall that a tree is a connected graph with no cycles. Show that every tree with n vertices has (exactly) $n-1$ edges.

2 Triangles in Graphs (Optional)

Theorem 1. *If a graph on $2n$ vertices has n^2+1 edges, then it has a triangle.*

Proof We prove it by induction on n. When $n=1$, the theorem is true, since the number of edges is at most $1 < n^2+1$.

In the general case, suppose the graph G has $2(n+1)$ vertices. Let $\{xy\}$ be an edge in the graph. Consider the graph G' on $2n$ vertices obtained by **deleting** x, y from the original graph. If G' has at least n^2+1 edges, then it has a triangle by induction, and we are done.

Otherwise, G' has at most n^2 edges. Since G has at least $(n+1)^2+1$ edges, by removing x, y we have deleted $(n+1)^2+1-n^2 = 2n+2$ edges from G. Since $\{x, y\}$ is also an edge, there are at least $2n+1$ edges that connect x, y to the vertices of G'. Thus by the pigeonhole principle, there is some vertex z so that $\{x, z\}, \{y, z\}$ are both edges. Then x, y, z form a triangle. ∎

The above theorem is tight. Consider the graph with n vertices on the left and n vertices on the right and every vertex on the left is connected to every vertex on the right. This graph has no triangles but n^2 edges.

Also, note the importance of deletion in the induction. Here, we crucially used that the x, y pair deleted from G were neighbors.