1 Asymptotics

Some properties of asymptotics:

- If \( f \leq O(g) \) and \( g \leq O(h) \) then \( f \leq O(h) \).
- If \( f \geq \Omega(g) \) and \( g \geq \Omega(h) \) then \( f \geq \Omega(h) \).
- If \( f = \Theta(g) \) and \( g = \Theta(h) \) then \( f = \Theta(h) \).
- If \( f = O(h) \), \( g = O(h) \) then \( f + g = O(h) \).

Some common running times:

- Polynomial: \( O(n^d) \). Exponential \( 2^{O(n)} \), Logarithmic \( O(\log n) \).
- For every positive \( \epsilon \) (no matter how small), \( \log n \leq O(n^\epsilon) \). For every positive \( d \) (no matter how large), \( n^d \leq O(2^n) \).

2 In class exercise

Arrange in increasing order of asymptotic growth. All logs are in base 2.

a) \( n^{5/3} \)

b) \( 2^{\sqrt{\log n}} \)

c) \( \sqrt{n^n} \)

d) \( \frac{n^2}{\log n} \)

e) \( 2^n \).

**Hint:** Recall rules of logarithm

- \( \log(a \cdot b) = \log a + \log b \),
- \( \log(a/b) = \log a - \log b \),
- \( \log a^b = b \log a \).

Always keep in mind \( n = 2^{\log_2 n} \). For example, \( n^{1.5} = 2^{1.5 \log_2 n} \). Also recall that \( (2^a)^b = 2^{a \cdot b} \).
3 Solution

In this part I will discuss the solution to the exercise. In many cases it might be difficult to directly compare two function \( f(n), g(n) \) asymptotically. An idea that usually helps out is to compare \( \log f(n) \) with \( \log g(n) \). Recall that logarithm is an increasing function, so if \( f(n) > g(n) \) for some \( n > N \) then \( \log f(n) > \log g(n) \) for \( n > N \). Here are two important rules when comparing the logs.

**When comparing logarithms ignore additive constants:** We said in class that \( n, 3n \) are asymptotically the same. If you take the log, then you are comparing \( \log n \) with \( \log n + \log 3 \). So, you can ignore the additive \( \log 3 \).

**When comparing logarithms, multiplicative constants matter:** Consider the two functions \( n, n^2 \). Obviously \( n^2 \) grows asymptotically faster. When comparing the log, we have \( \log n, 2 \log n \). So, the 2 multiplicative constant matters and shows that \( n^2 \) grows faster.

Having said these, I will write down the solution to exercise. First, we calculate the logs:

a) \( \log n^{5/3} = \frac{5}{3} \log n \).

b) \( \log 2^{\sqrt{\log n}} = \sqrt{\log n} \log 2 = \sqrt{\log n} \).

c) \( \log \sqrt{n^2} = \frac{1}{2} \log n^n = \frac{n}{2} \log n \).

d) \( \log \frac{n^2}{\log n} = \log n^2 - \log \log n = 2 \log n - \log \log n \).

e) \( \log 2^n = n \log 2 = n \).

Therefore, \( \sqrt{\log n} < \frac{5}{3} \log n < 2 \log n - \log \log n < n < \frac{n}{2} \log n \). Note that when comparing \( \frac{5}{3} \log n \) and \( 2 \log n - \log \log n \), \( \log \log n \) is a lower order term. So, first we compare the dominating terms \( 5/3 \log n \) and \( 2 \log n \). In this case the latter is bigger. If the dominating term were the same then we would have compared to lower order terms.