Claim $G=(X,Y)$, $|X|=|Y|$. If $|N(S)| \geq |S|$ for all $S \subseteq X$ then, G has a perfect matching.

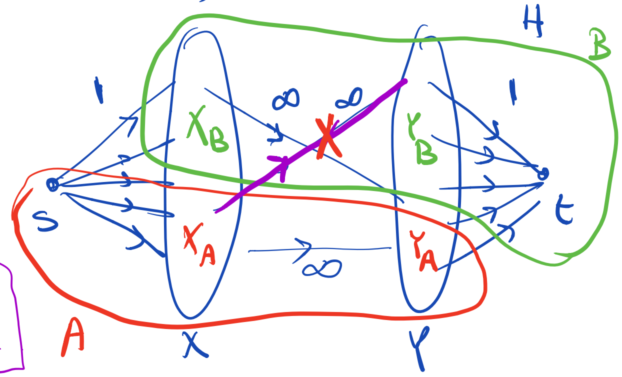
Pf: (By contradiction) Supp G has no Perfect matching.
We will find some $S \subseteq X$ s.t. $|N(S)| < |S|$.

$|X| > \text{Max Matching } G = \text{max flow}(H) = \text{min cut}(H)$

Let (A,B) be min s-t cut of H

$s \in A$, $t \in B$, $X_A = X \cap A$, $Y_A = Y \cap A$, ...

$\text{cap}(A,B) < |X|$. (*)



Claim: No edge $X_A \to Y_B$
any such edge has infinit cap
B/C cap(A,B) is finite, it does not exist

$\text{cap}(A,B) = |X_B| + |Y_A|$ (**)

$$|N(X_A)| \underset{\substack{\text{B/C} \\ \text{no edge } X_A \to Y_B}}{\leq} |Y_A| \underset{(**)}{=} \text{cap}(A,B) - X_B \underset{(*)}{<} |X| - |X_B| = |X_A|$$

In HW8-P1 (you will do similar arg)