# CSE 421

## Bellman-Ford ALG, Network Flows

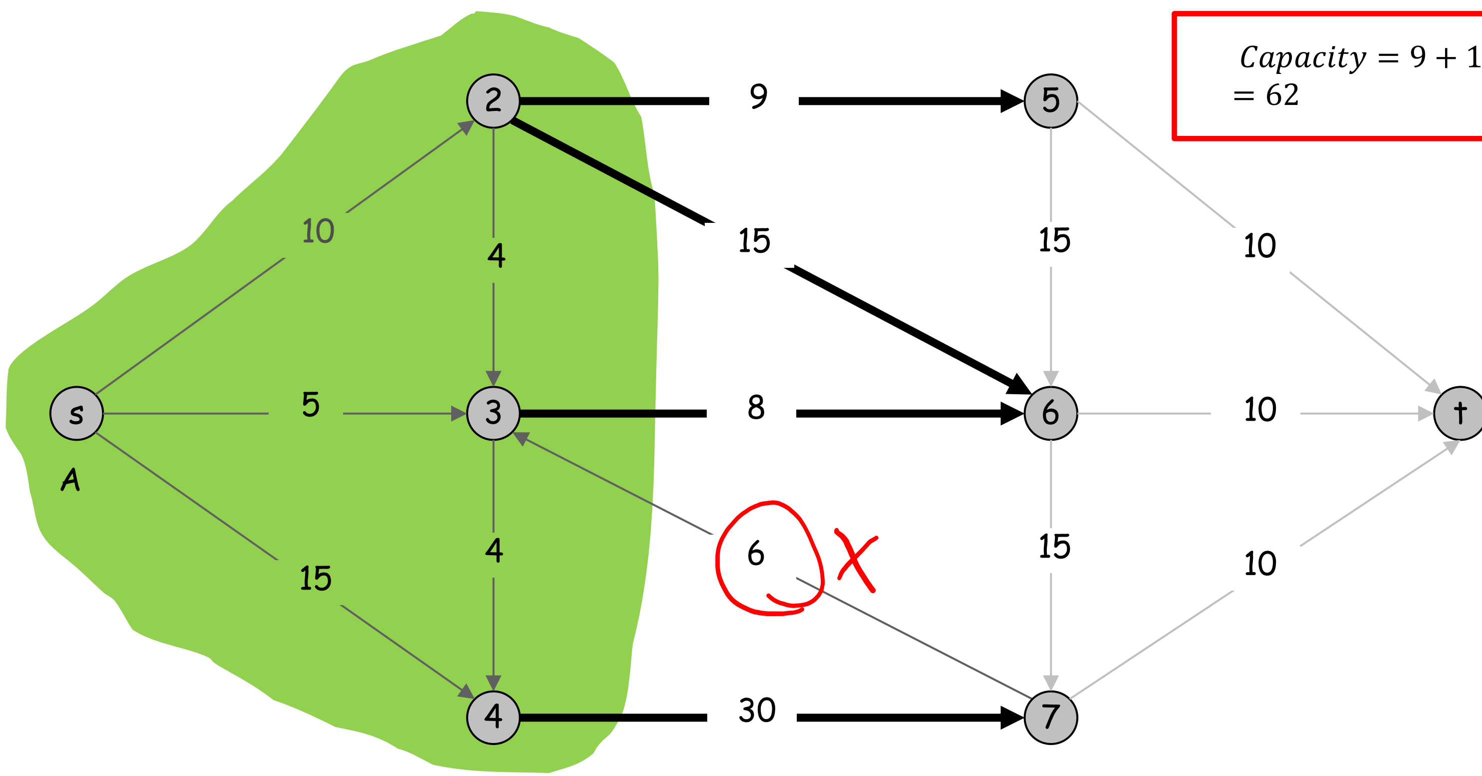
Shayan Oveis Gharan

# Network Flows

# s-t cuts

Def. An s-t cut is a partition (A, B) of V with s $\in$ A and t $\in$ B.

Def. The capacity of a cut (A, B): $cap(A,B) = \sum_{(u,v):u\in A, v\in B} c(u,v)$



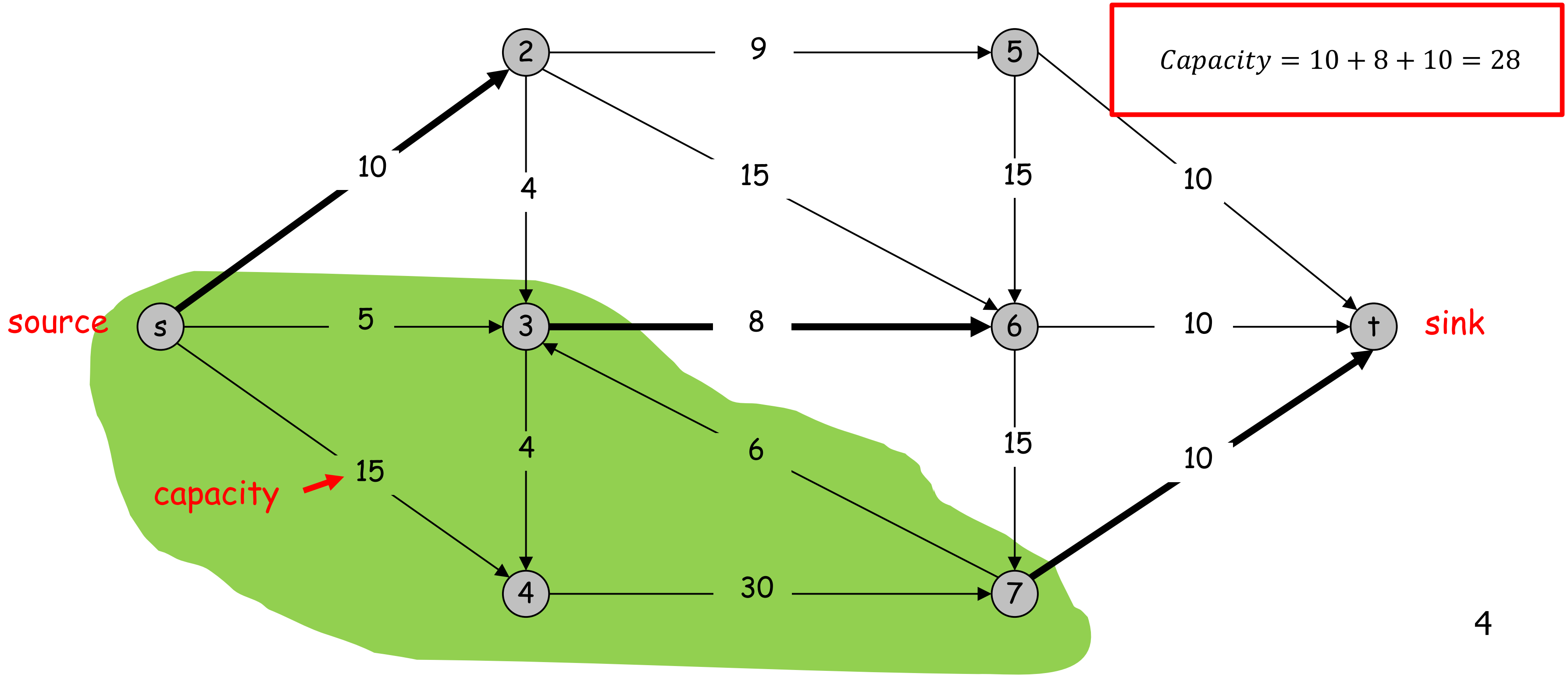

$Capacity = 9 + 15 + 8 + 30$
$= 62$

# Minimum s-t Cut Problem

Given a directed graph G = (V, E) = directed graph and two distinguished nodes:  s = source, t = sink.

Suppose each directed edge e has a nonnegative capacity $c(e)$

Goal: Find a s-t cut of minimum capacity

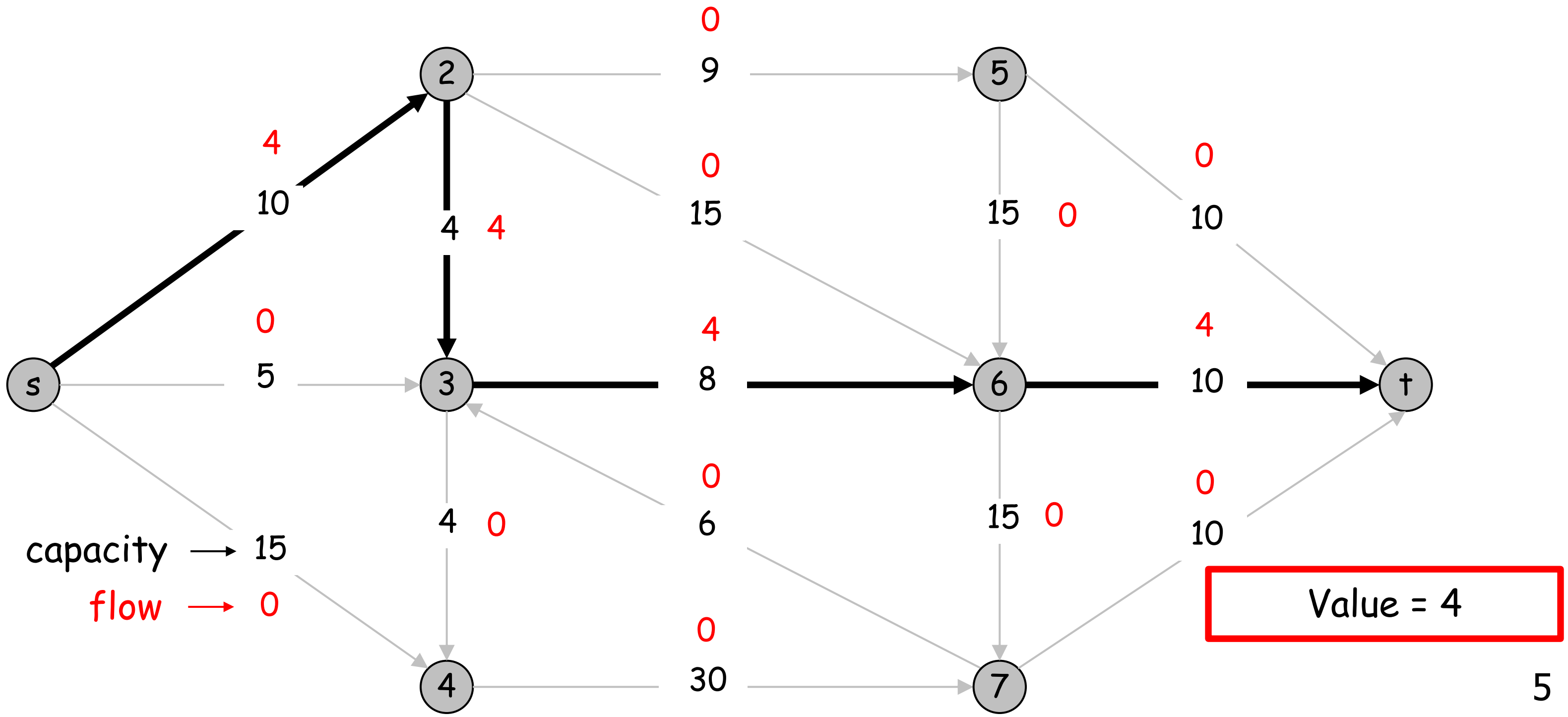


# s-t Flows

Def. An s-t flow is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

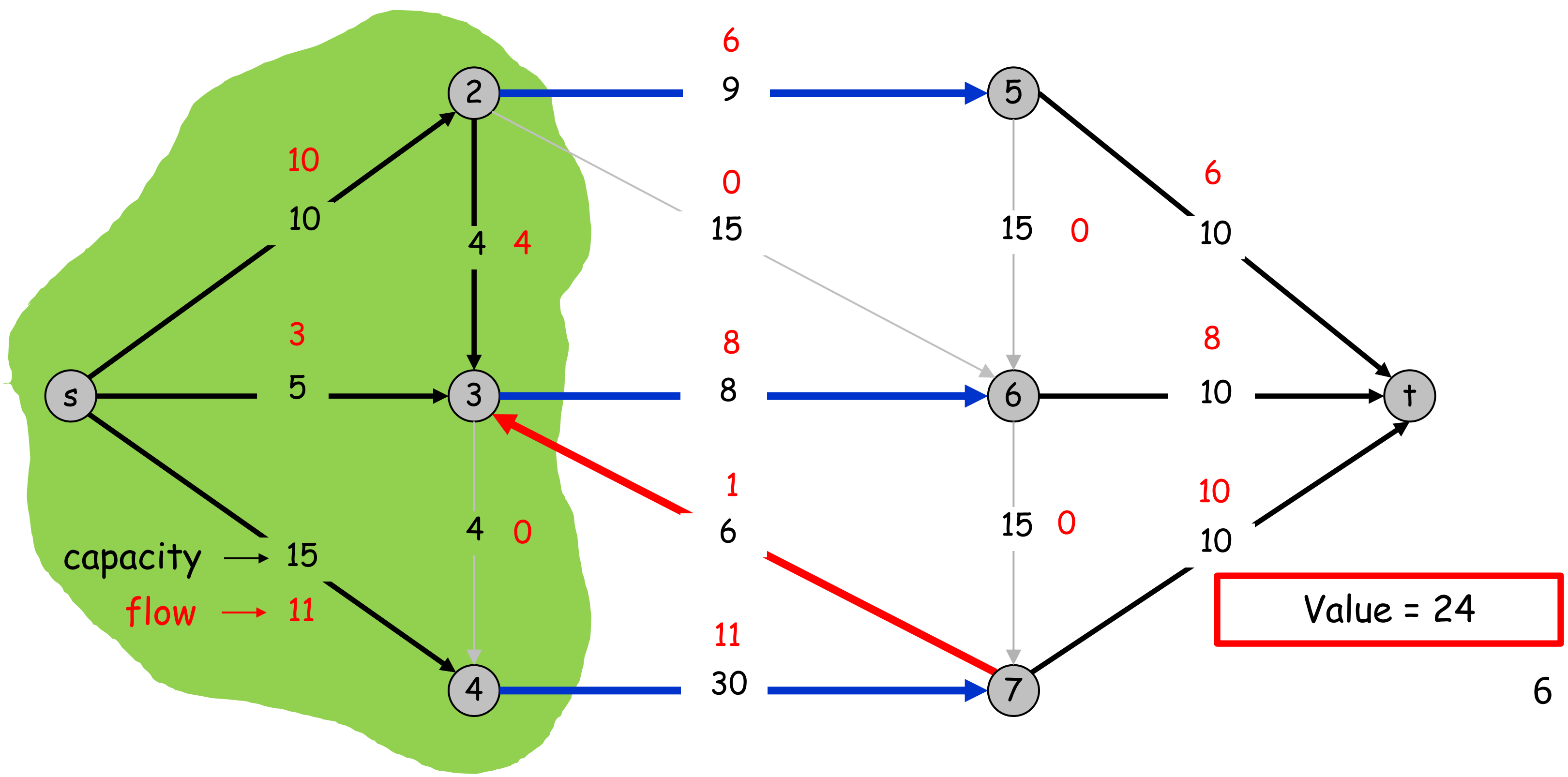
Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$



# Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



# Pf of Flow value Lemma

**Flow value lemma.** Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

By conservation of flow, all terms except v=s are0

$$= \sum_{v \in A} \left( \sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

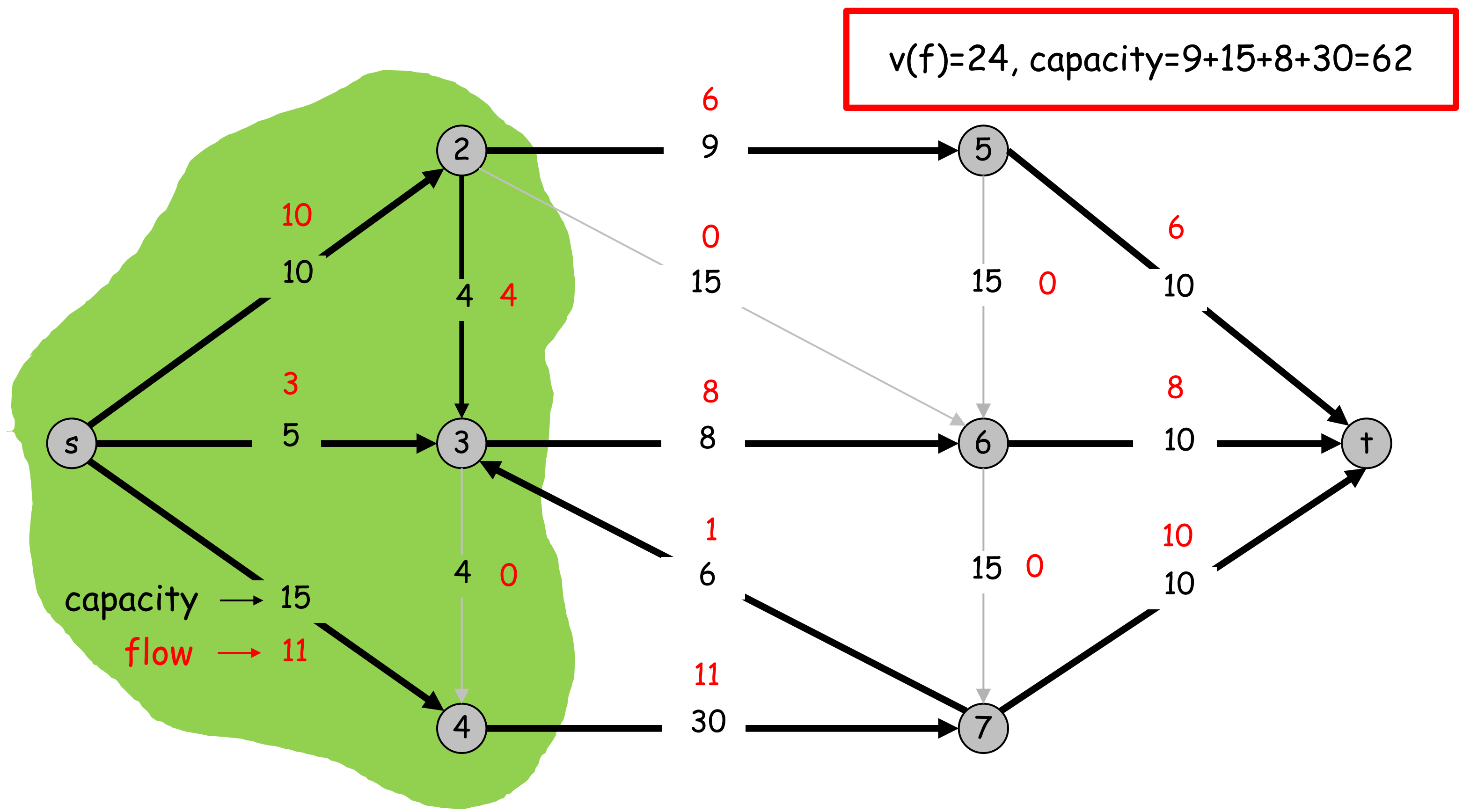
All contributions due to internal edges cancel out

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

# Weak Duality of Flows and Cuts

Cut Capacity lemma. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

$$v(f) \leq cap(A, B)$$



# Weak Duality of Flows and Cuts

Cut capacity lemma. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

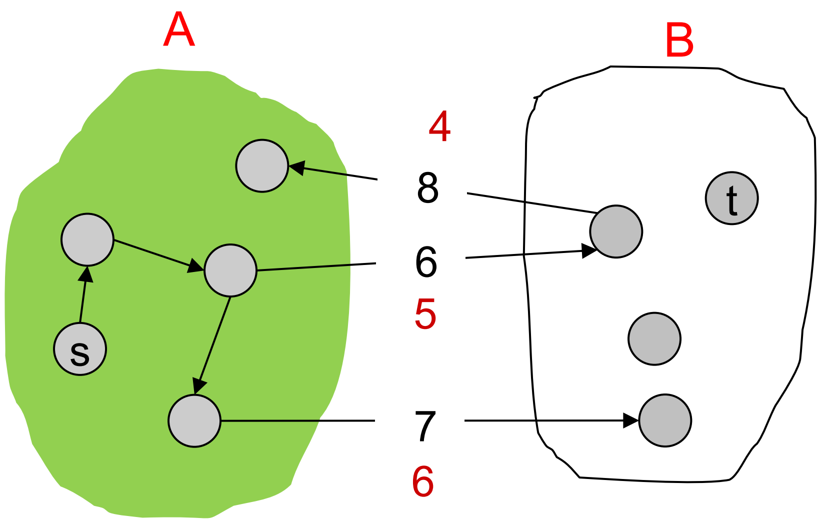
$$v(f) \leq cap(A, B)$$

Pf.

$$v(f) = \sum_{e\ out\ of\ A} f(e) - \sum_{e\ in\ to\ A} f(e)$$

$$\leq \sum_{e\ out\ of\ A} f(e)$$

$$\leq \sum_{e\ out\ of\ A} c(e) = cap(A, B)$$

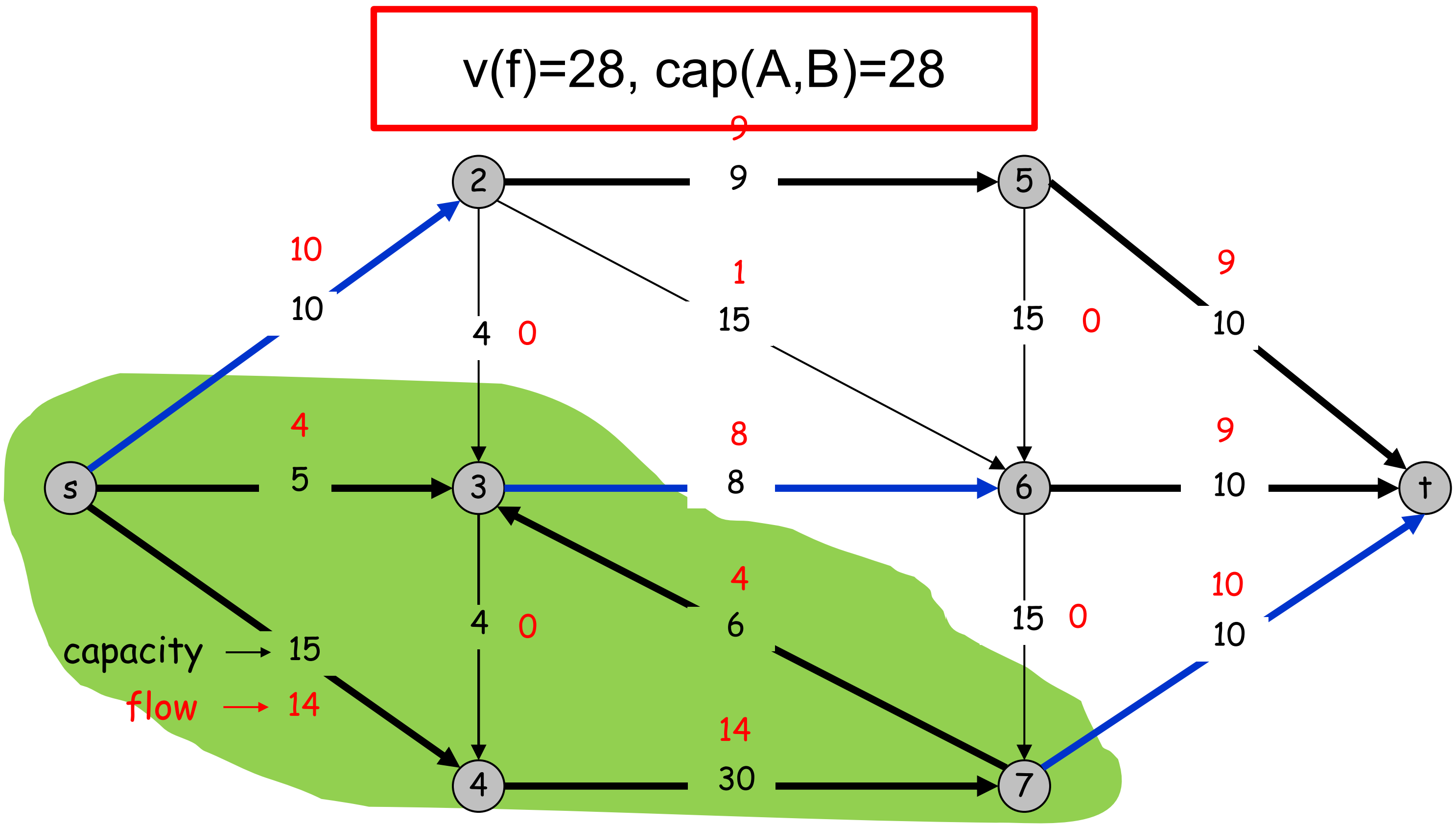


# Certificate of Optimality

Corollary: Suppose there is a s-t cut (A,B) such that

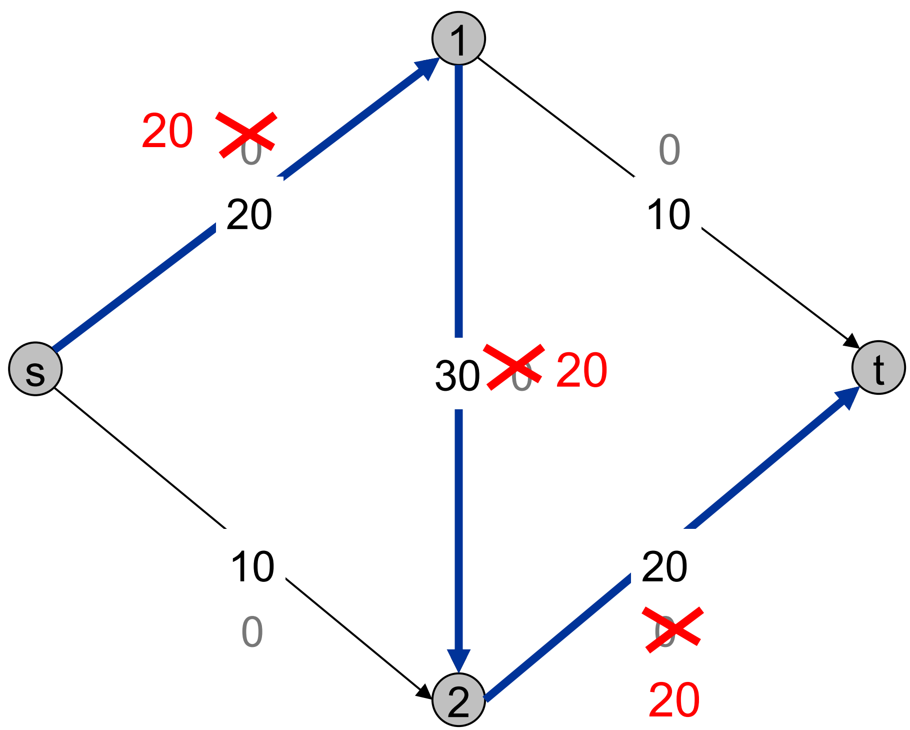
$$v(f) = cap(A, B)$$

Then, f is a maximum flow and (A,B) is a minimum cut.



# A Greedy Algorithm for Max Flow

- Start with f(e) = 0 for all edge e $\in$ E.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.



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- Start with f(e) = 0 for all edge e $\in$ E.
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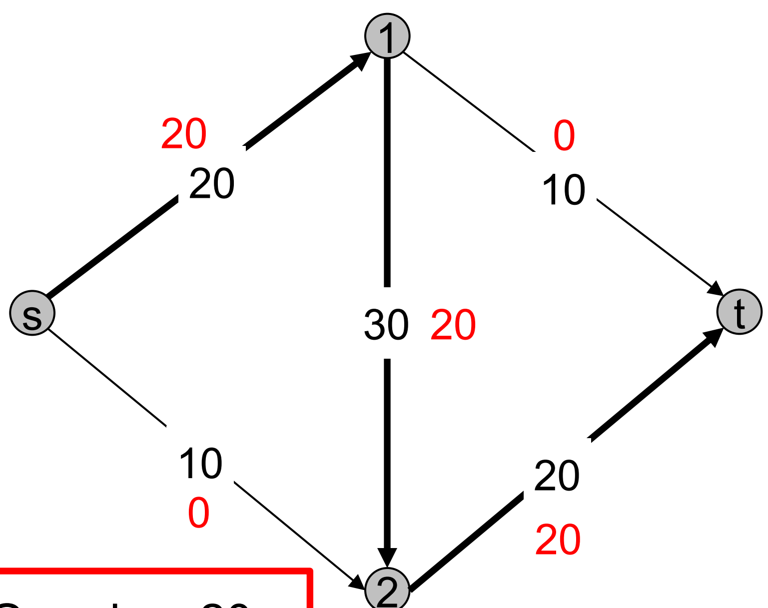
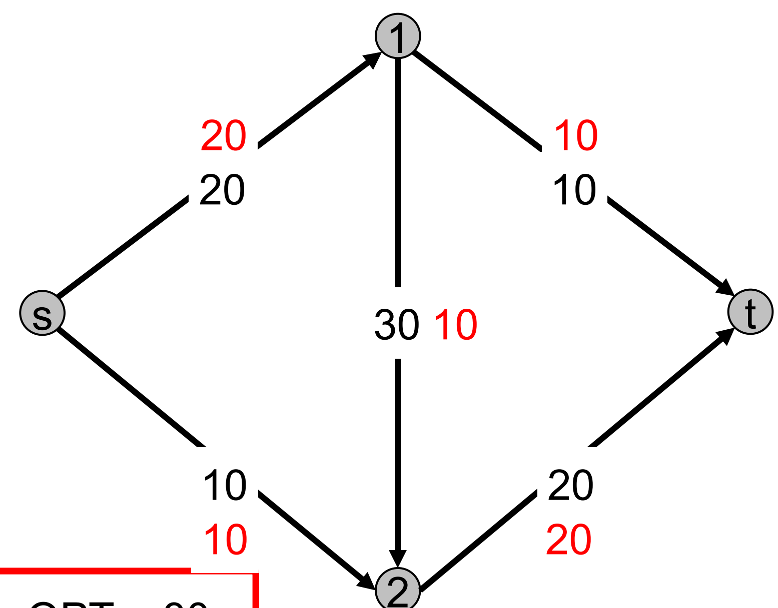
Local Optimum $\neq$ Global Optimum

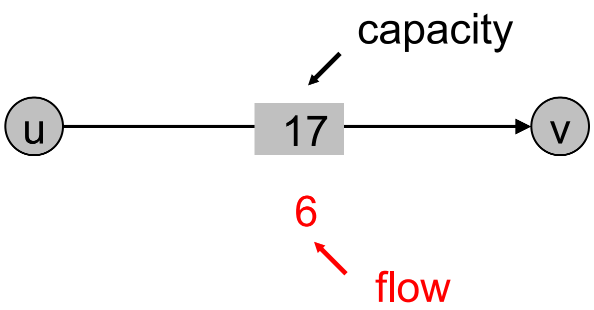



Greedy = 20




OPT = 30

# Residual Graph



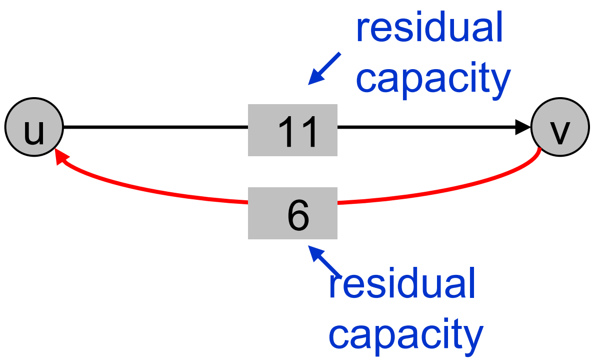

Original edge: e = (u, v) $\in$ E.

- Flow f(e), capacity c(e).

Residual edge.

- "Undo" flow sent.
- e = (u, v) and $e^R$ = (v, u).
- Residual capacity:

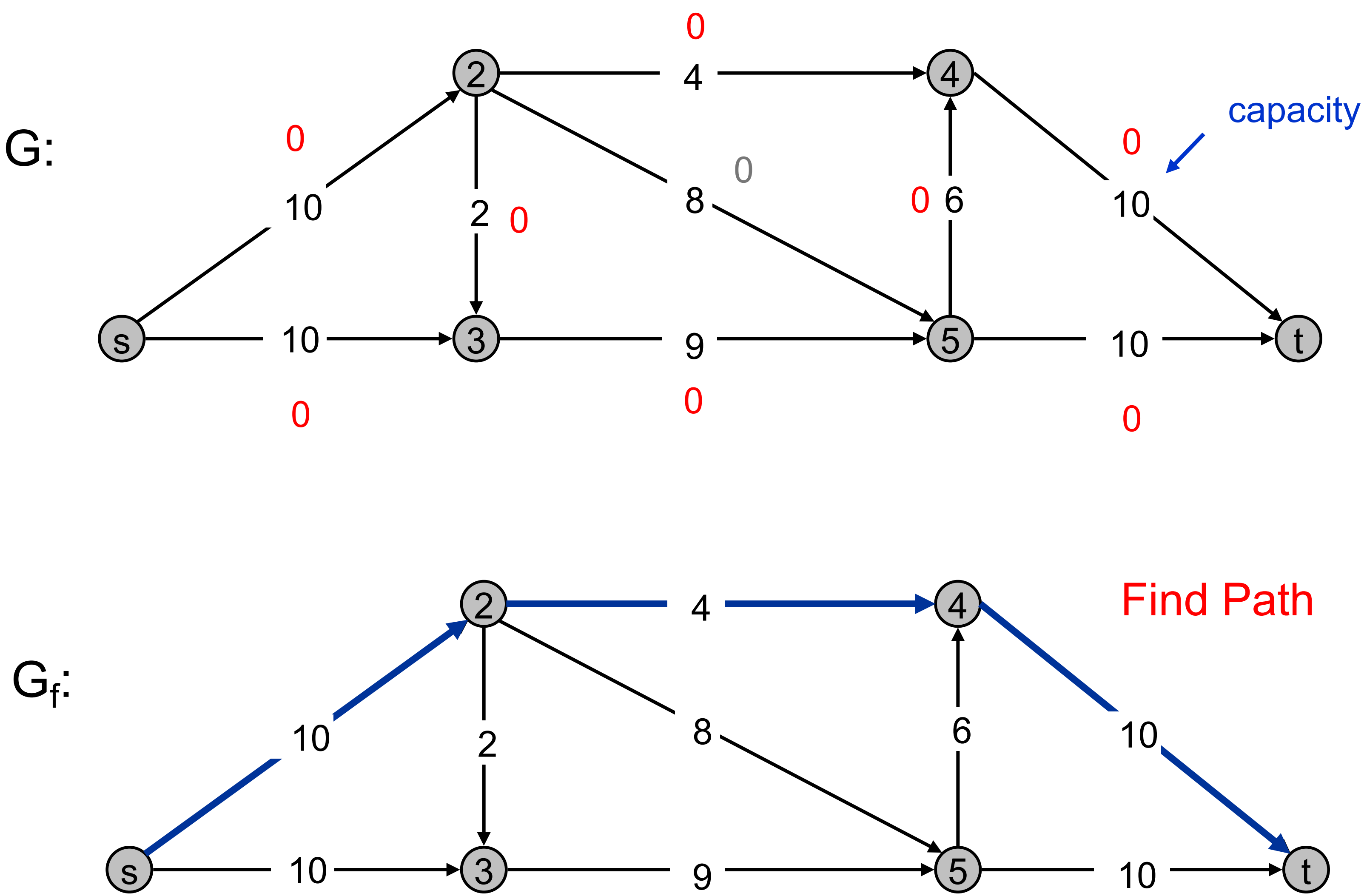
$$c_f(e) = \begin{cases} c(e) - f(e) & if\ e \in E \\ f(e) & if\ e^R \in E \end{cases}$$




Residual graph: $G_f = (V, E_f)$.

- Residual edges with positive residual capacity.
- $E_f \ = \{e : f(e) < \ c(e)\} \cup \{e : \ f(e^R) \ > \ 0\}$.

# Ford-Fulkerson Alg: Greedy on $G_f$



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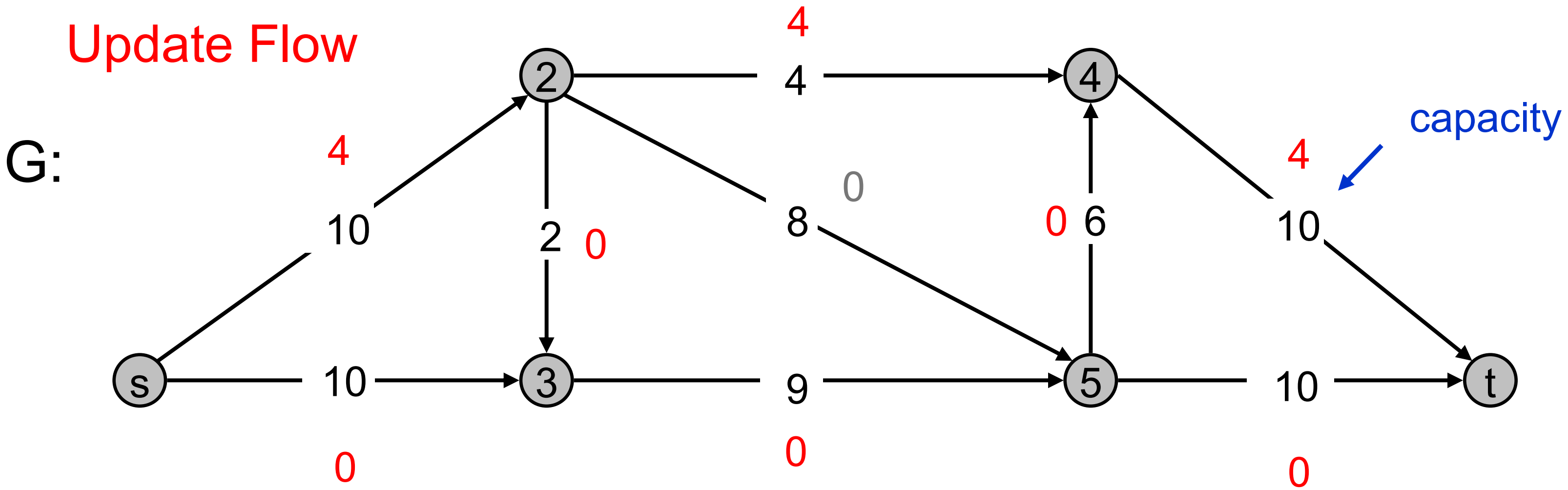


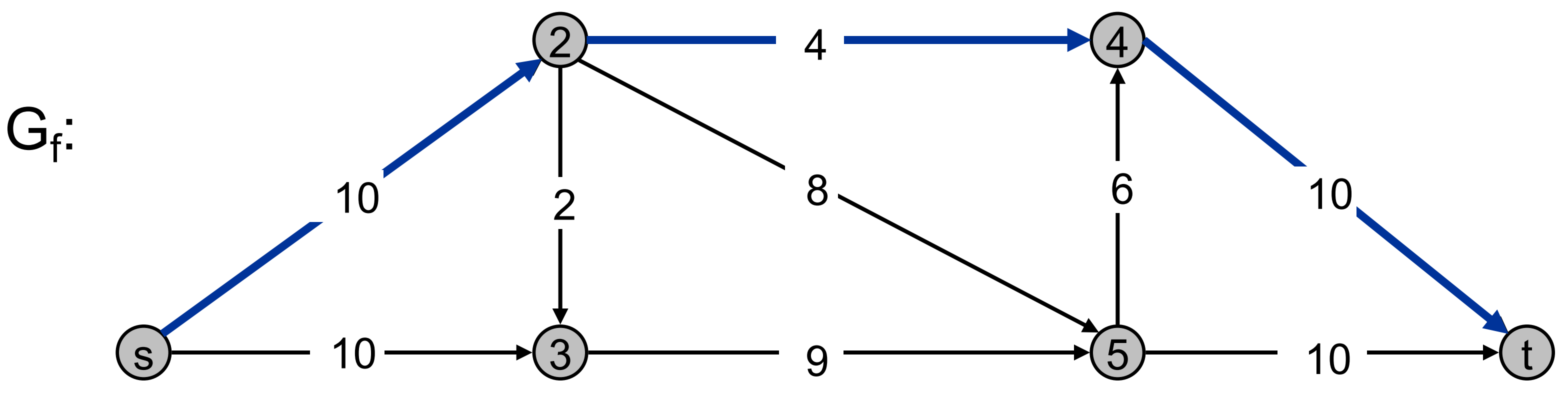

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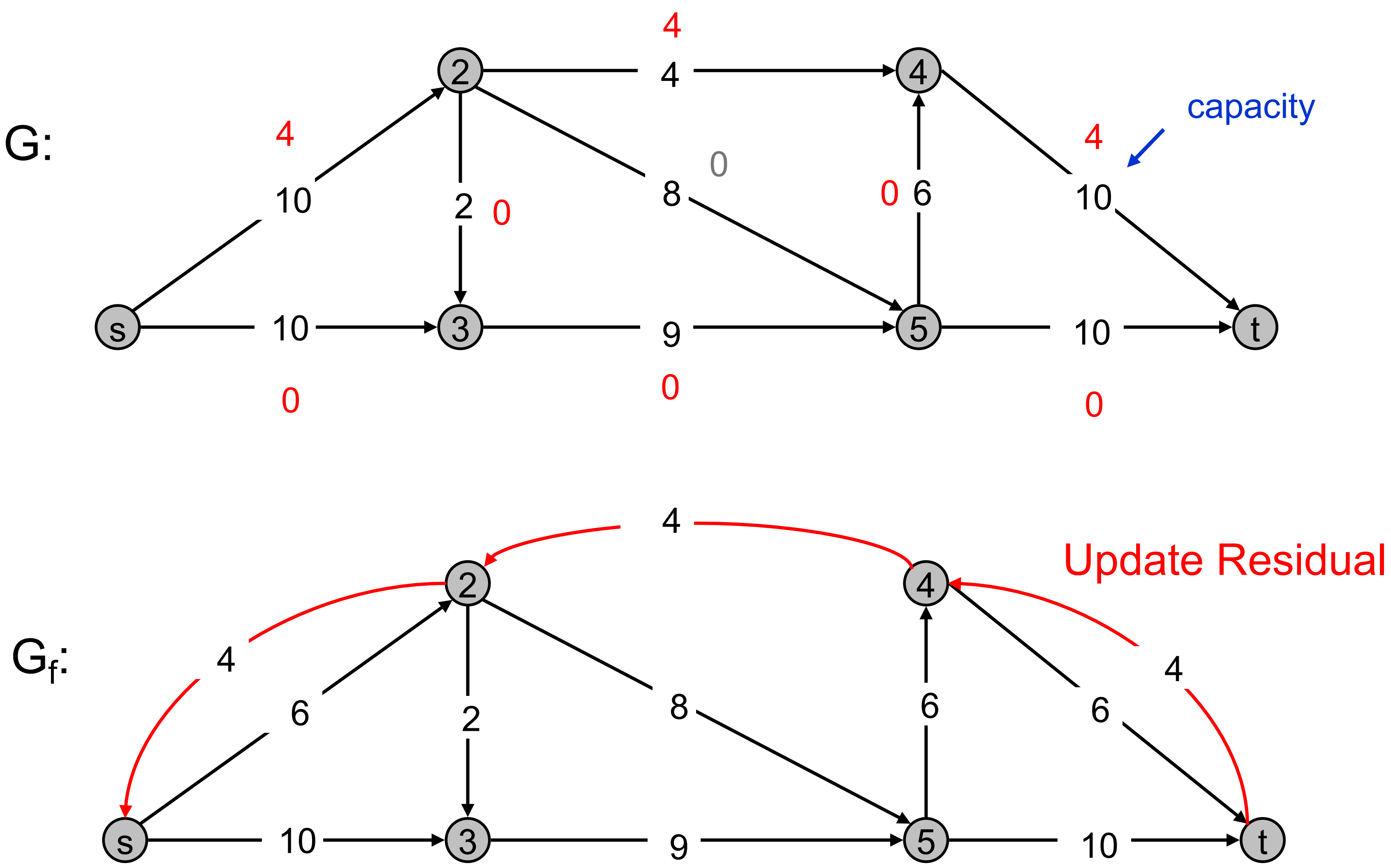

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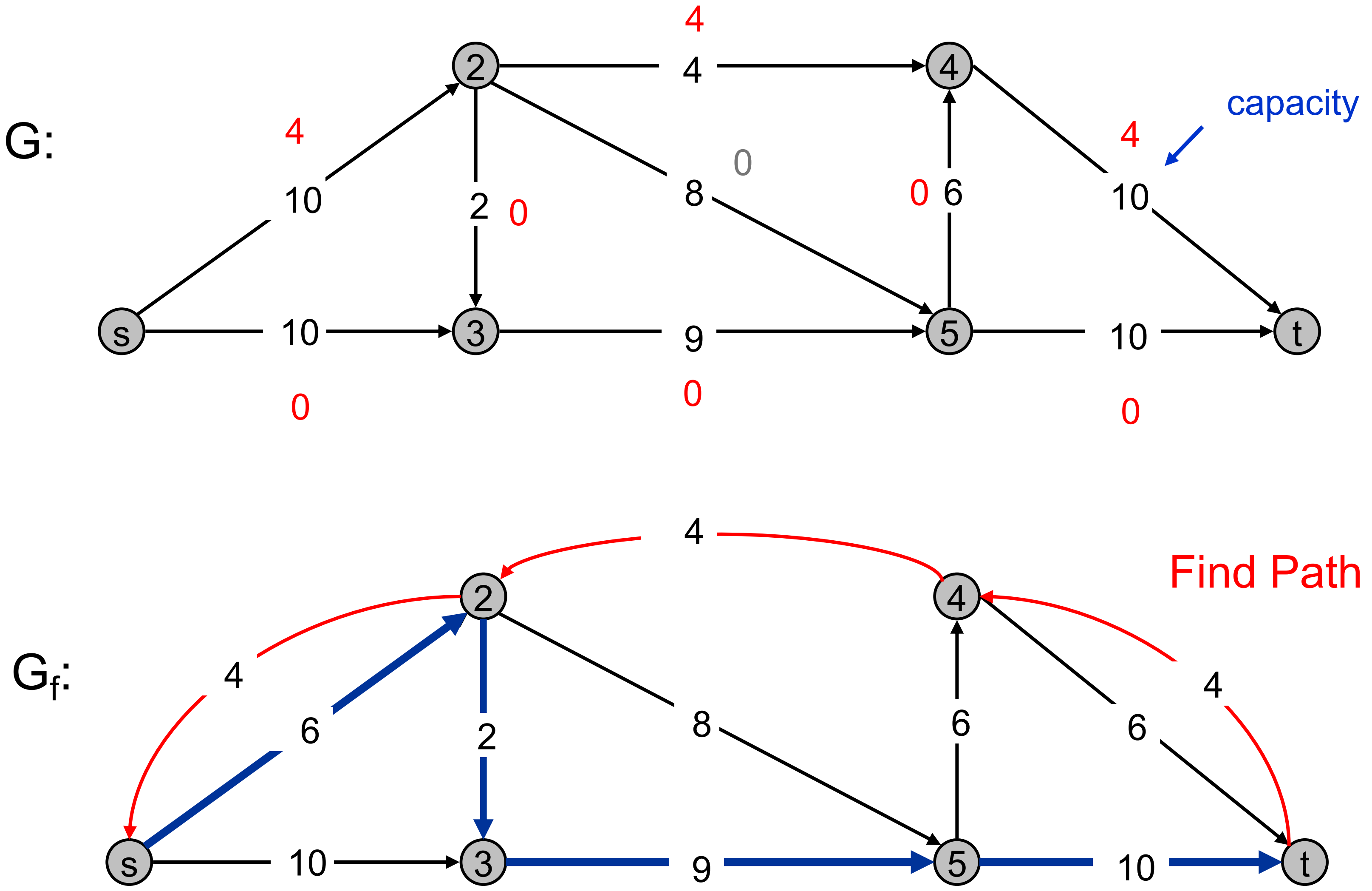

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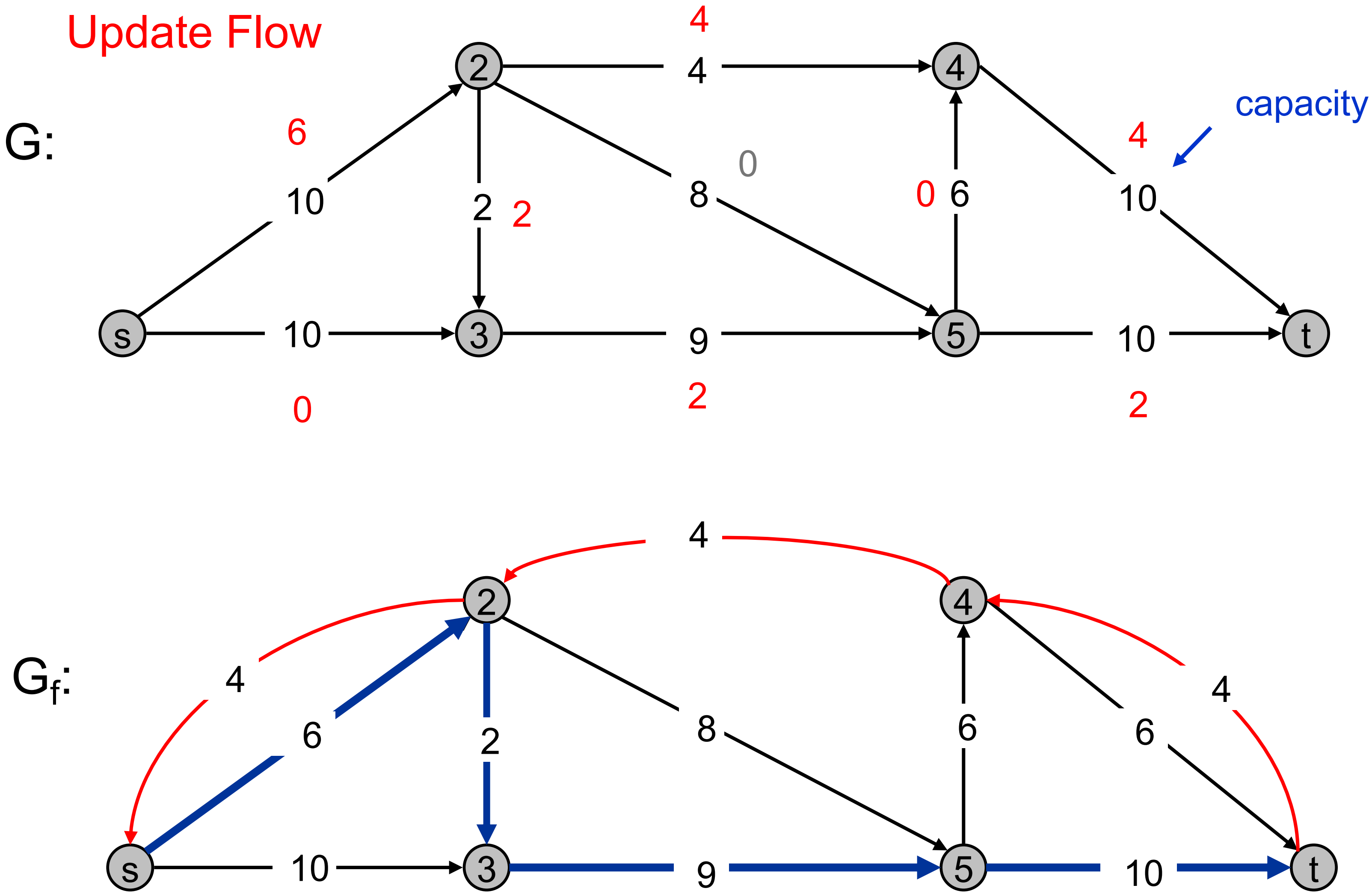

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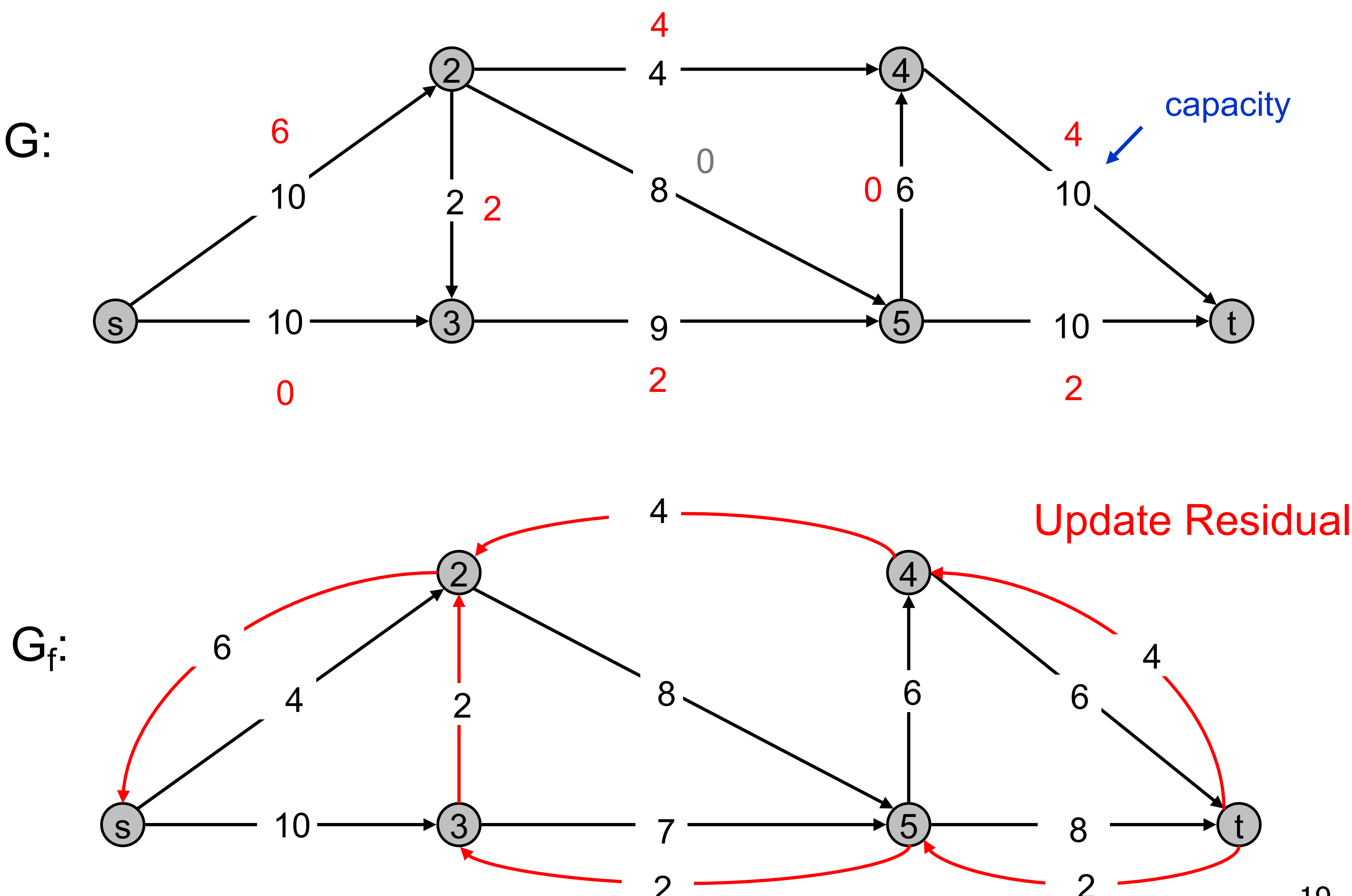

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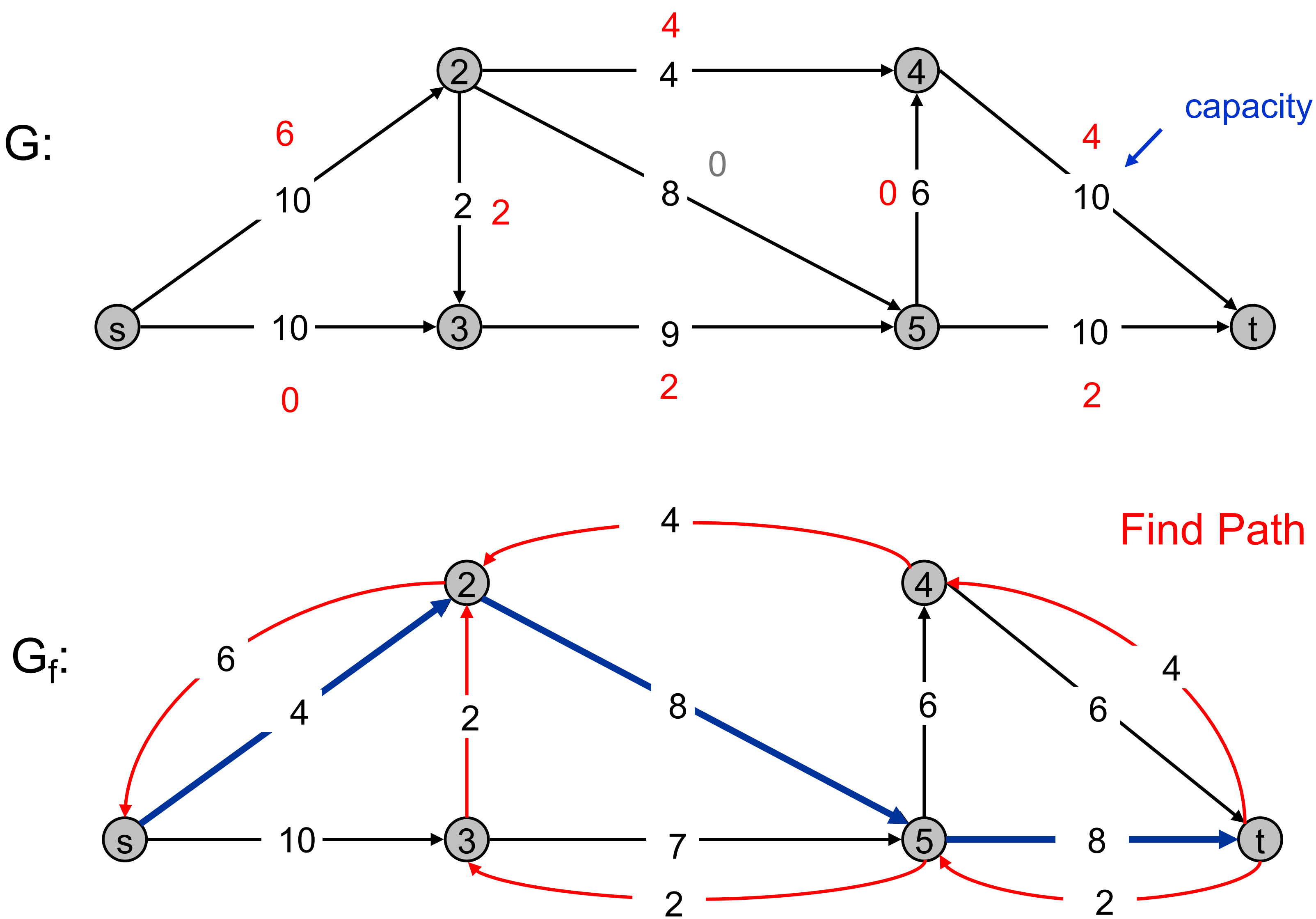

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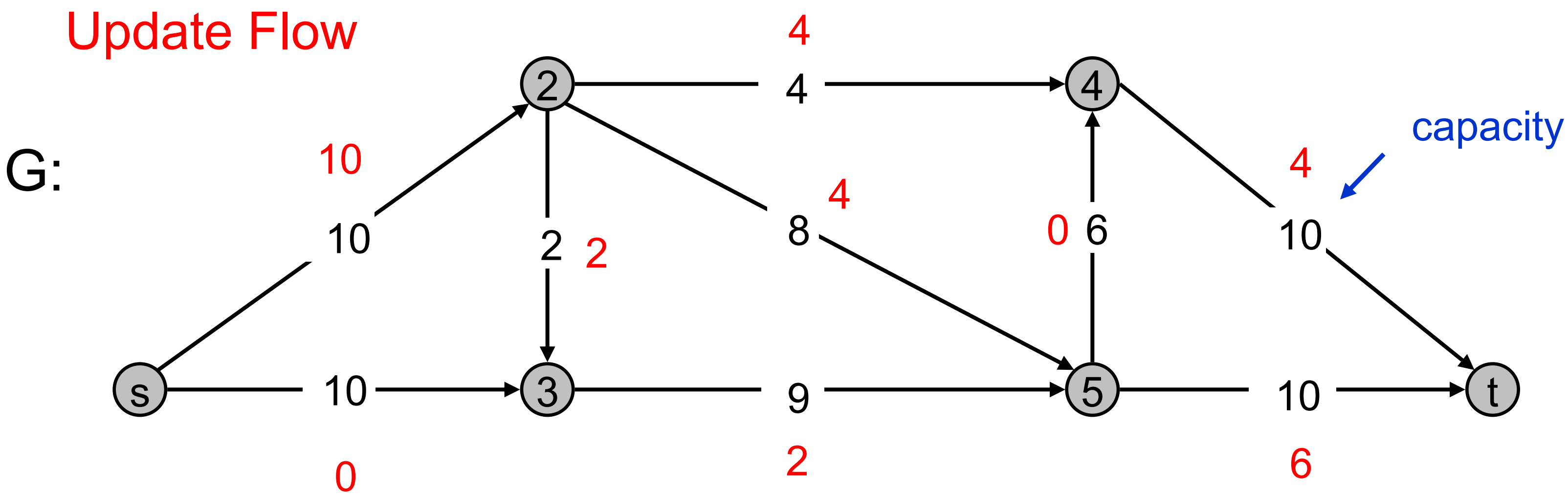


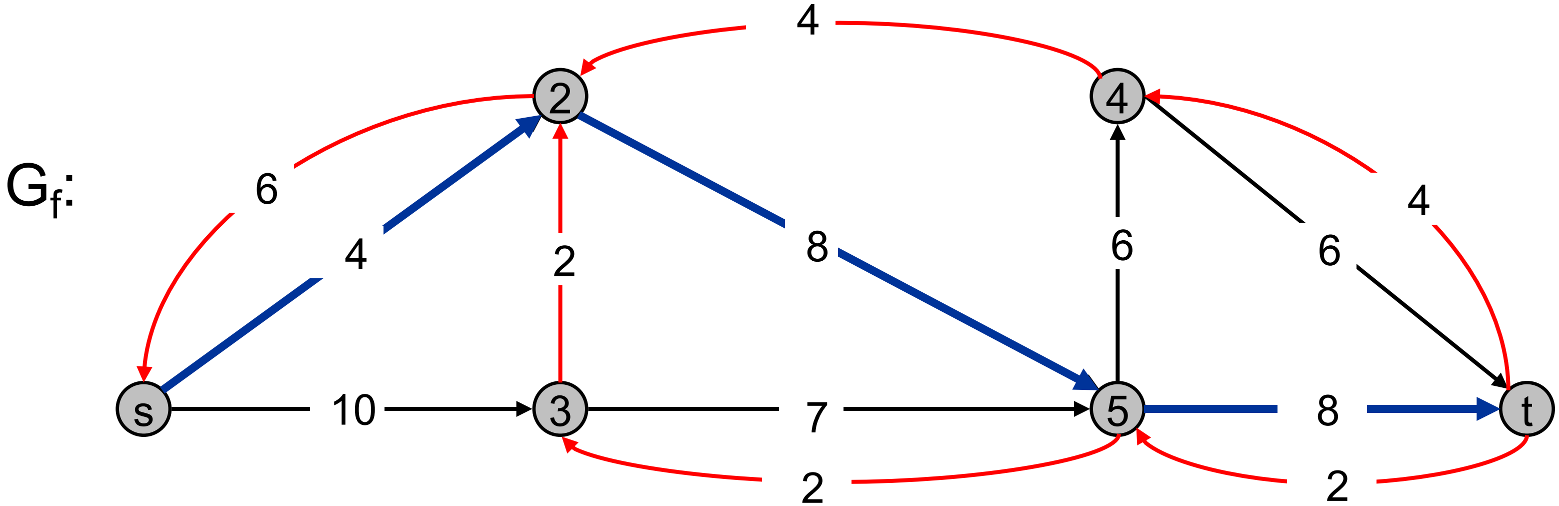

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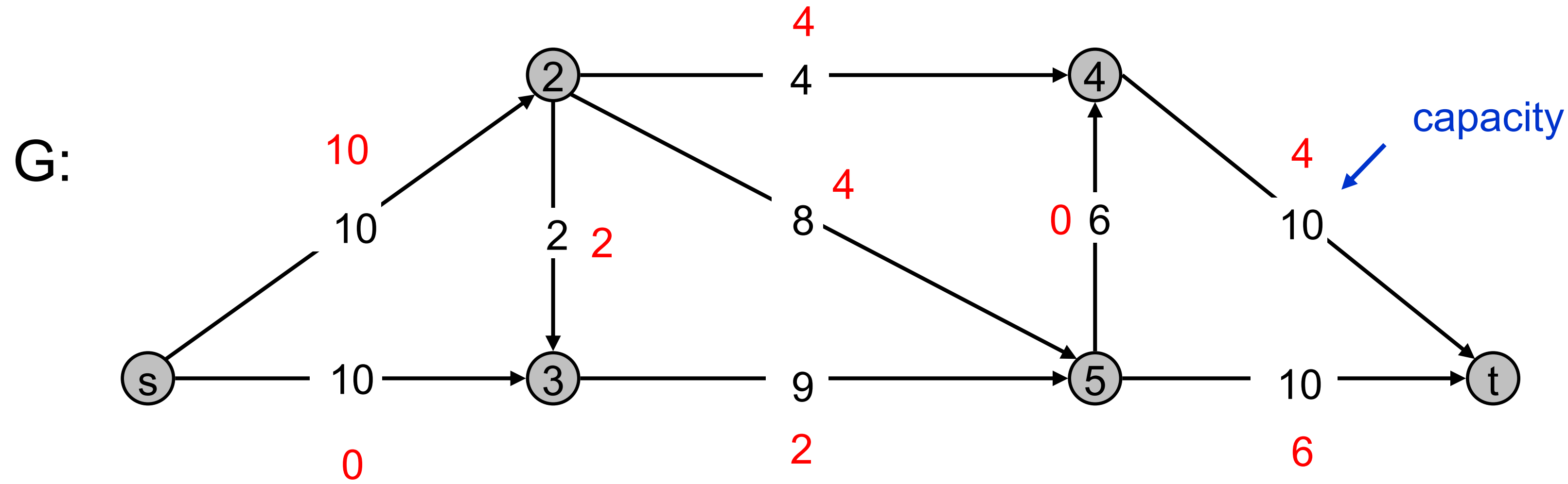
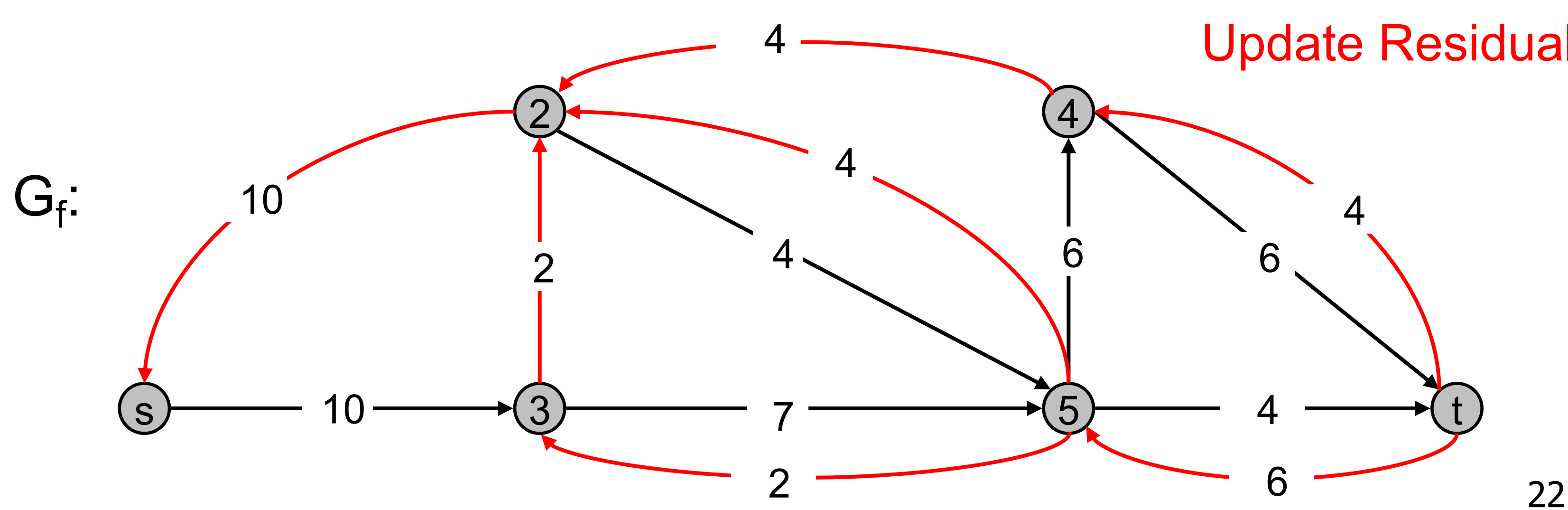
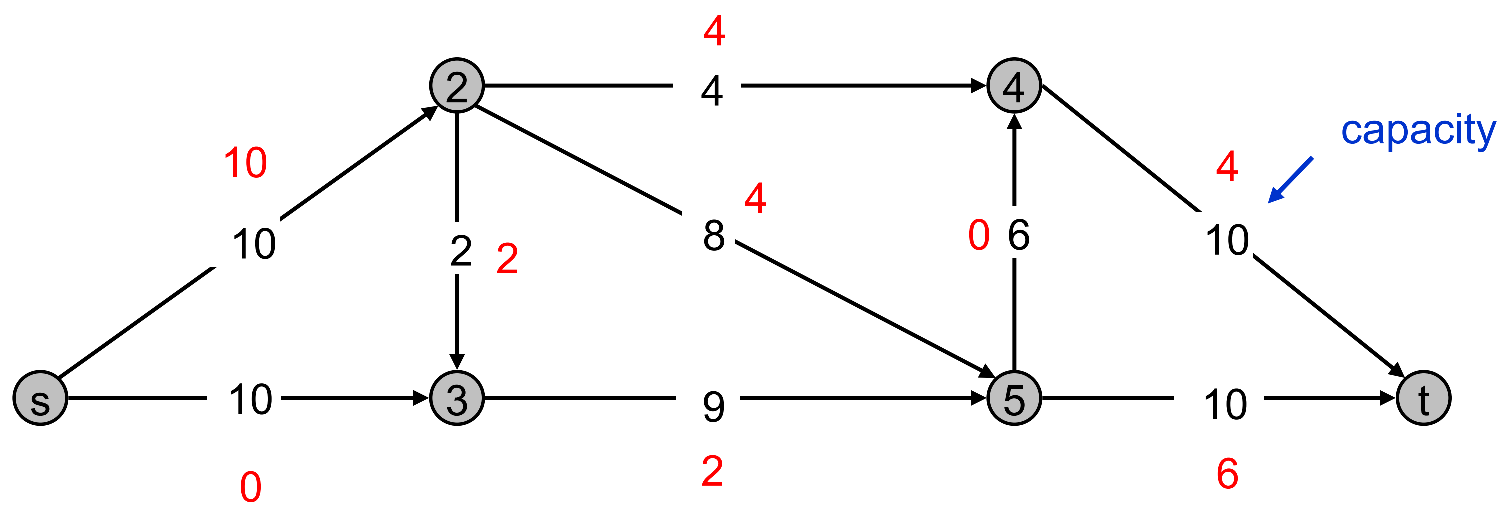


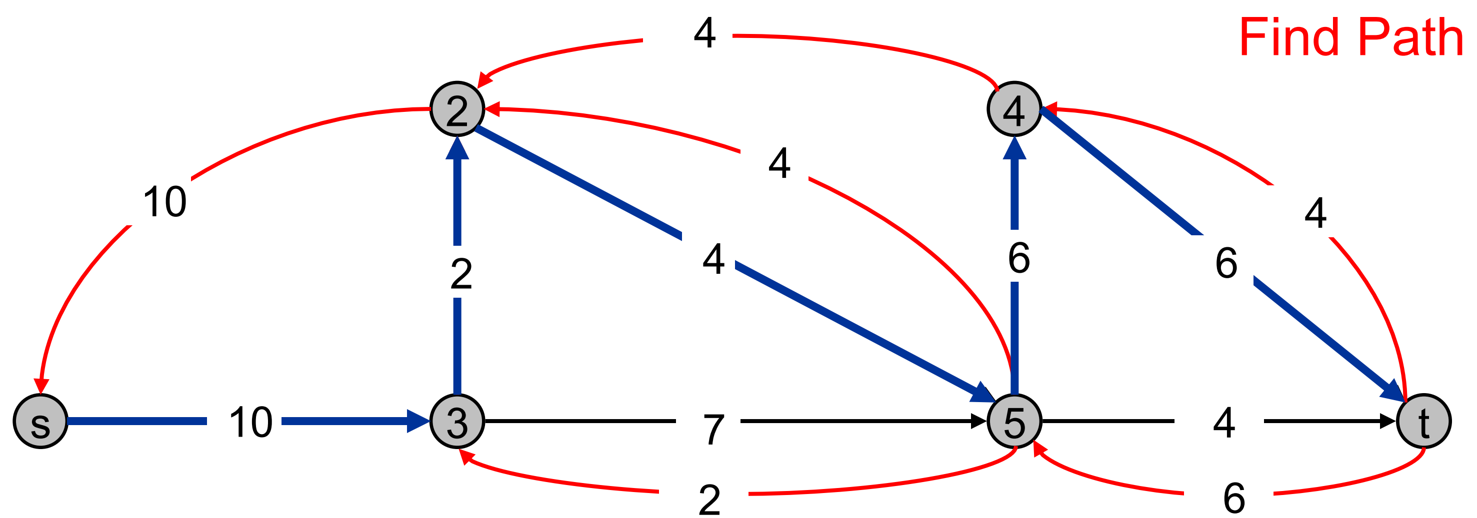



# Ford-Fulkerson Alg: Greedy on $G_f$

G:




$G_f$:



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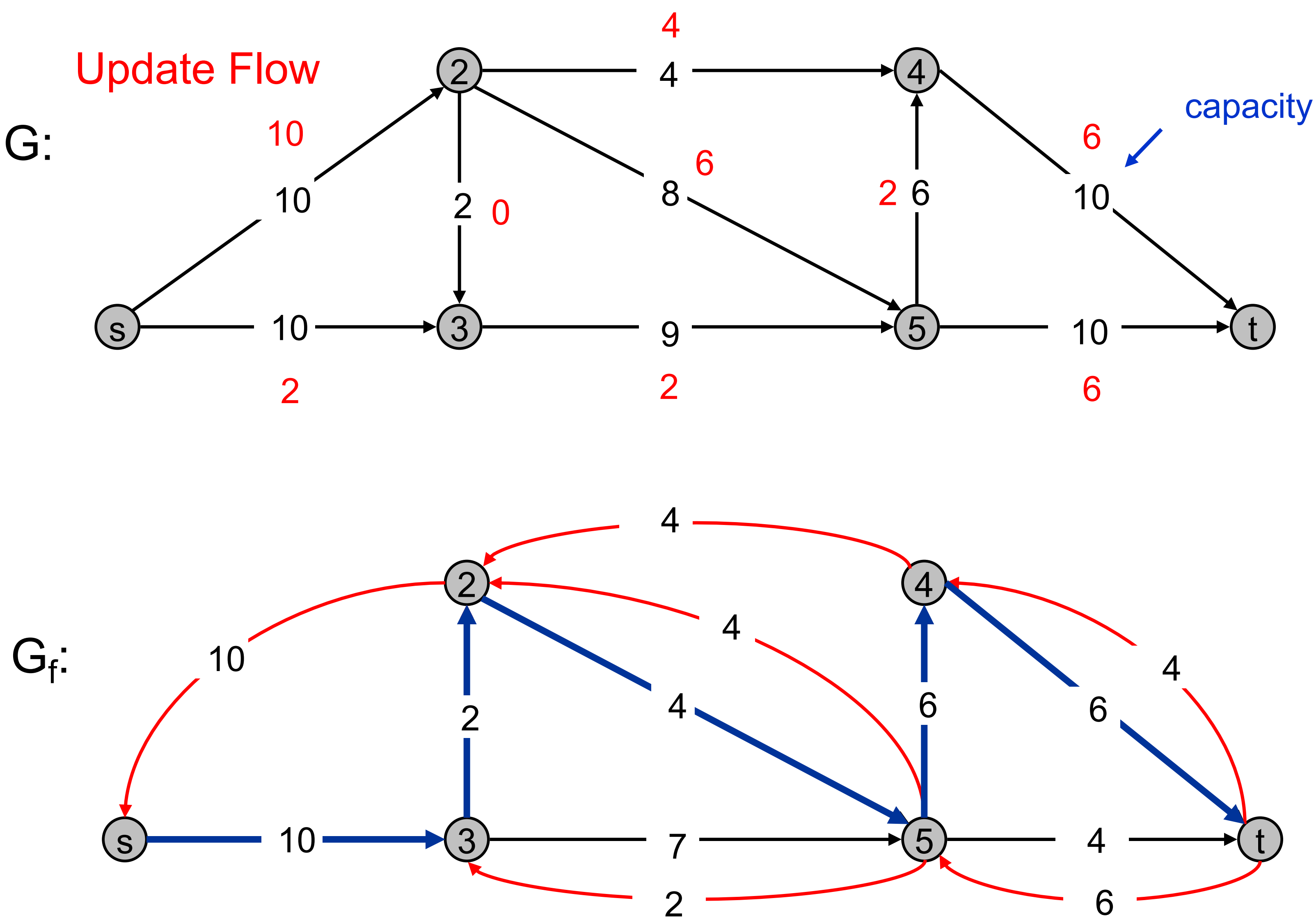

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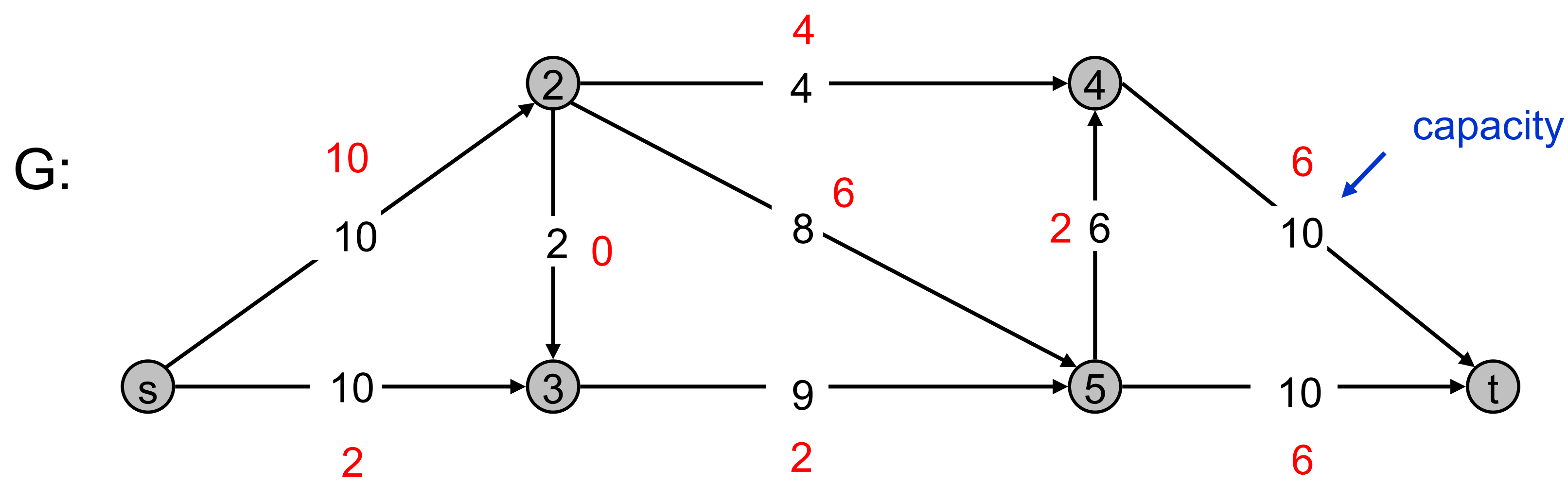


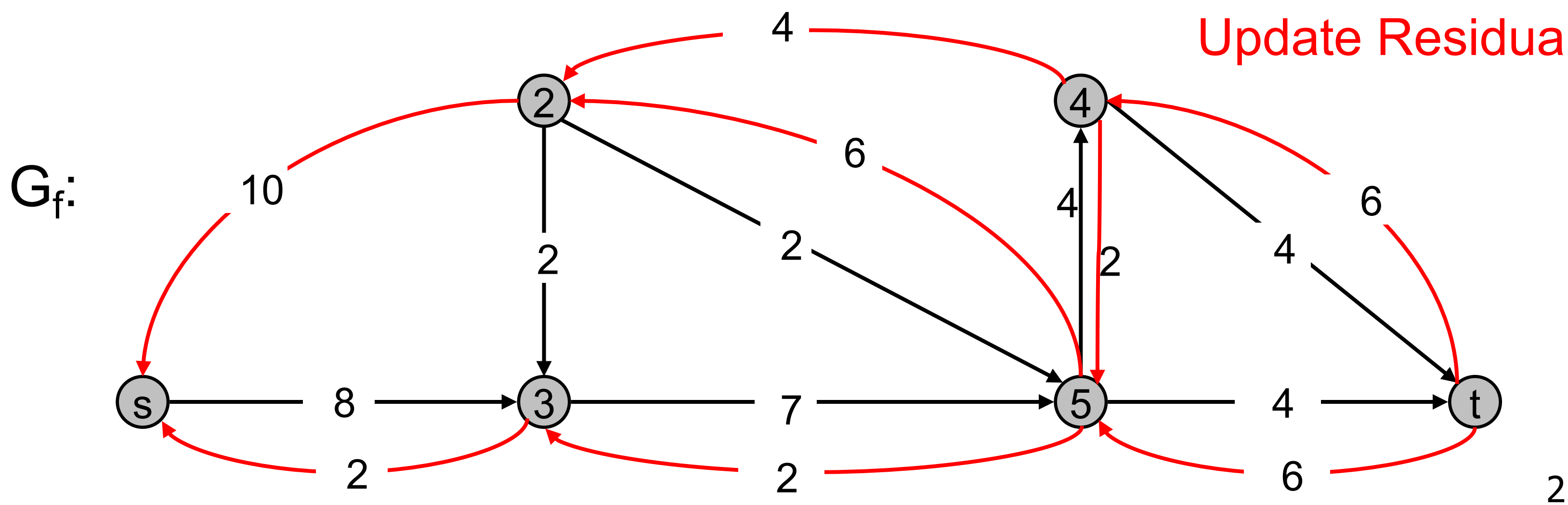

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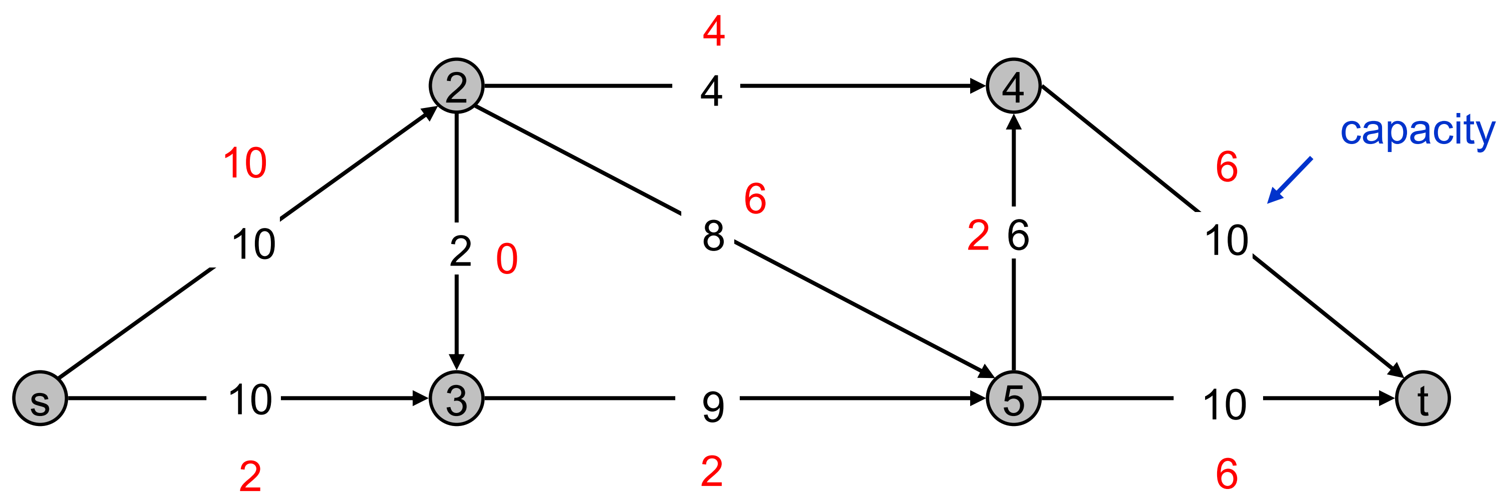
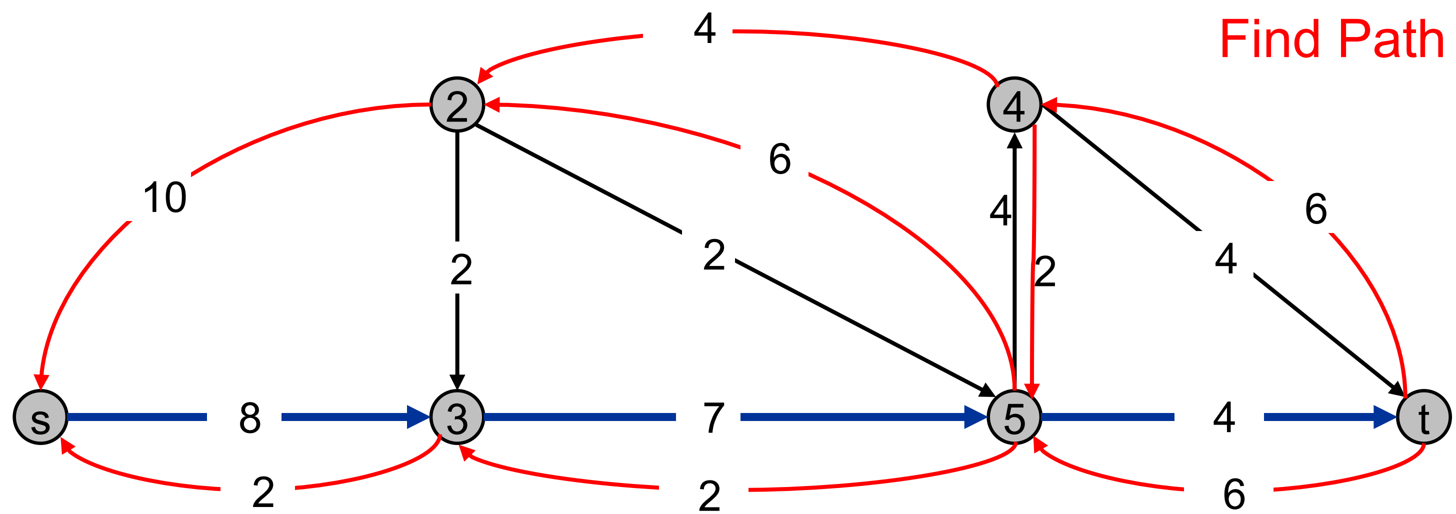



$G_f$:



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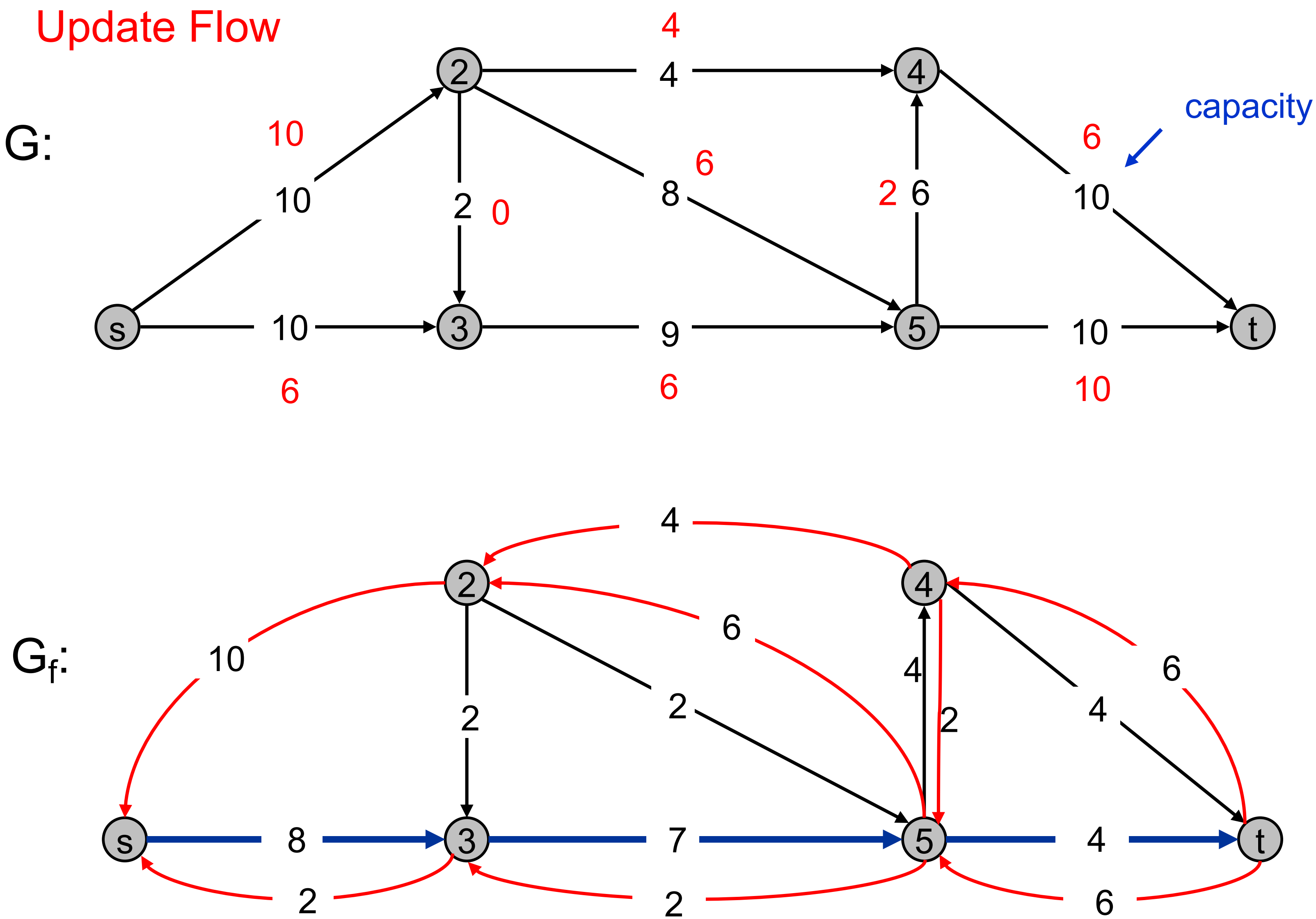

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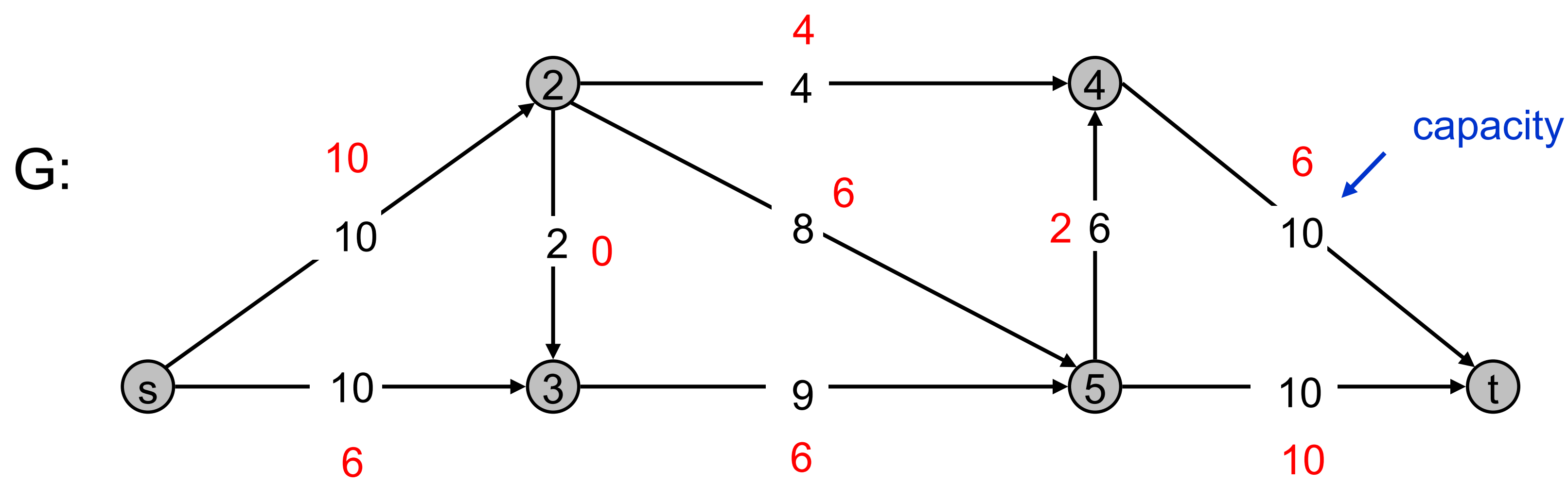


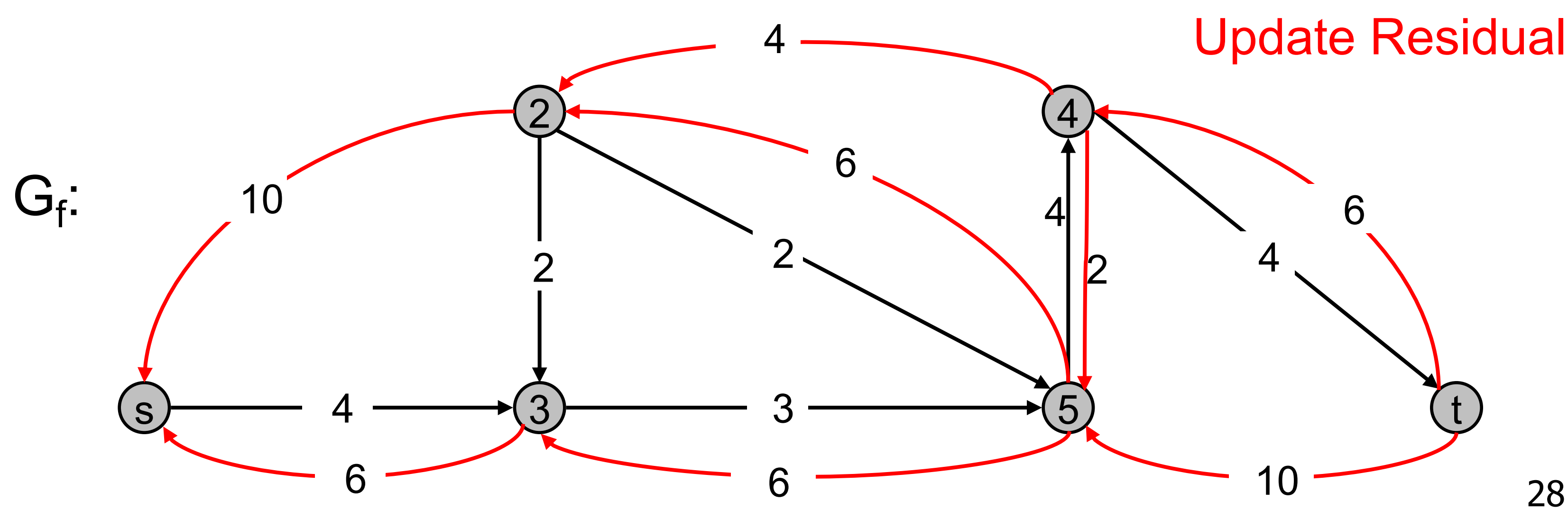

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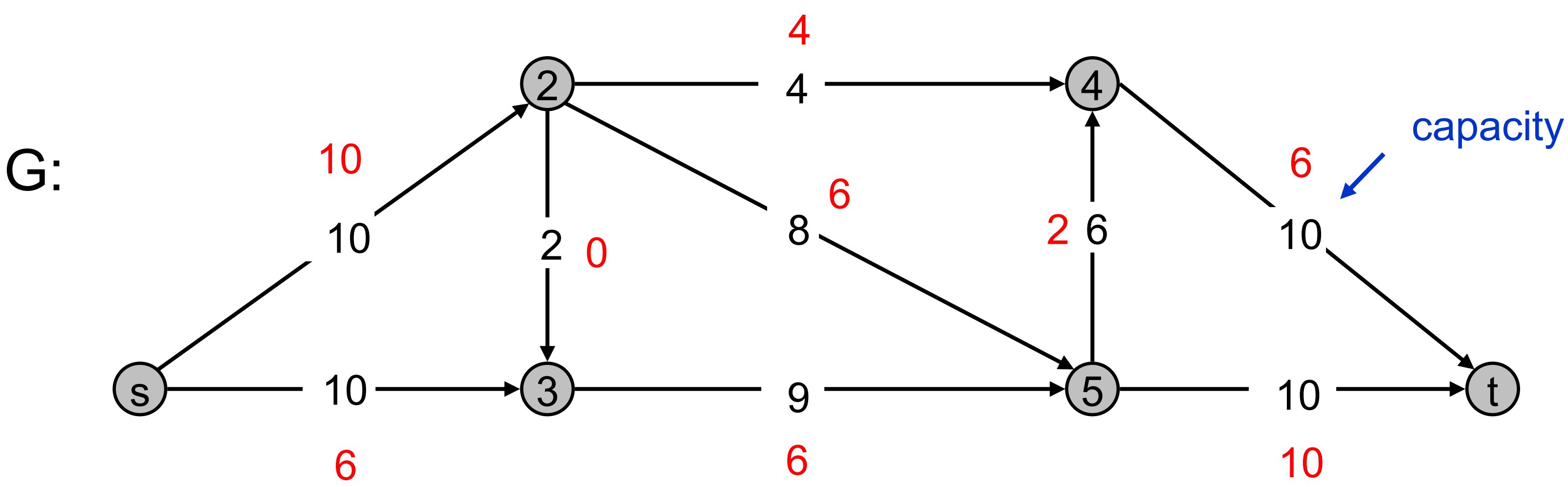


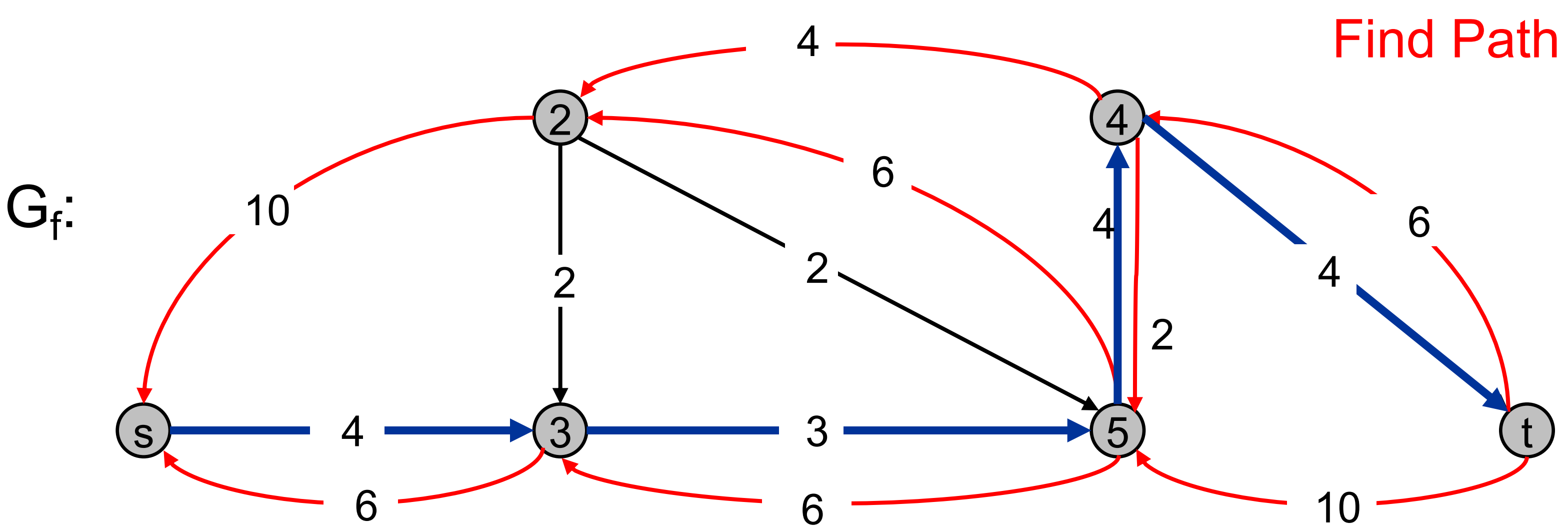

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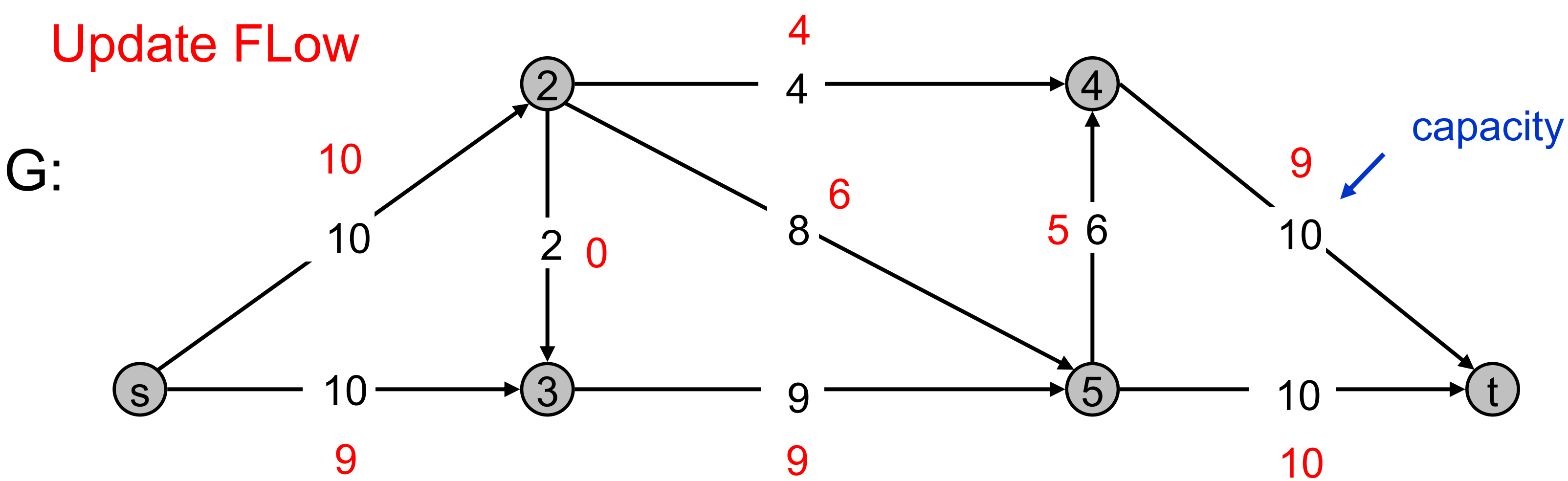


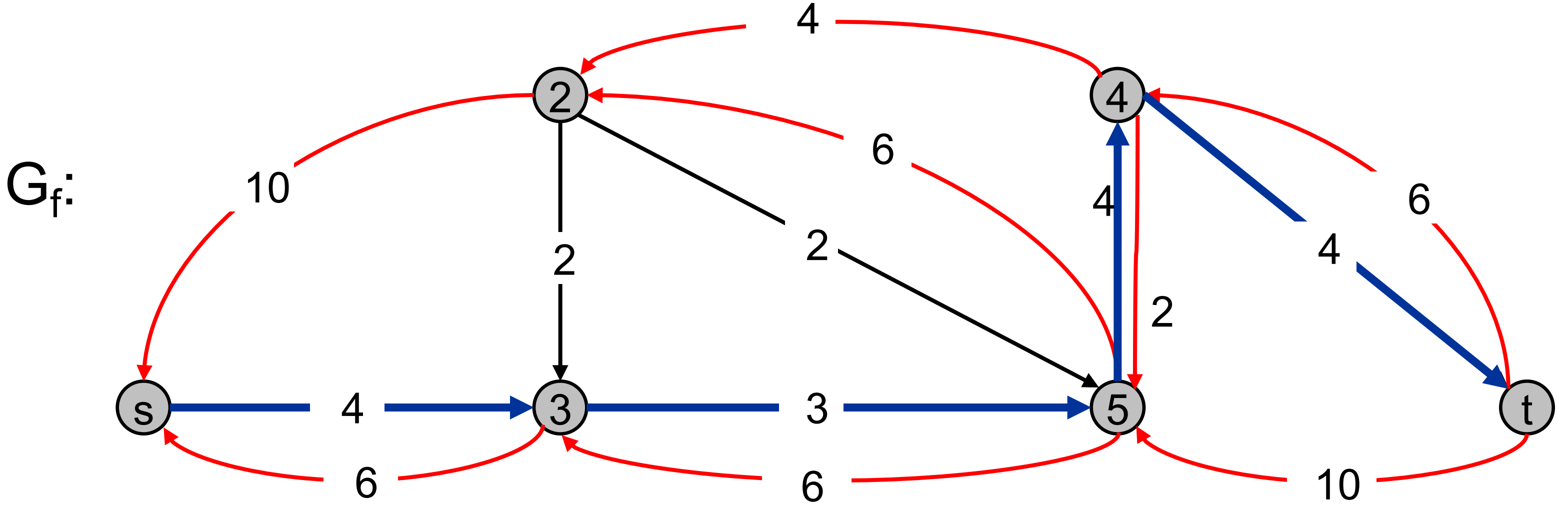

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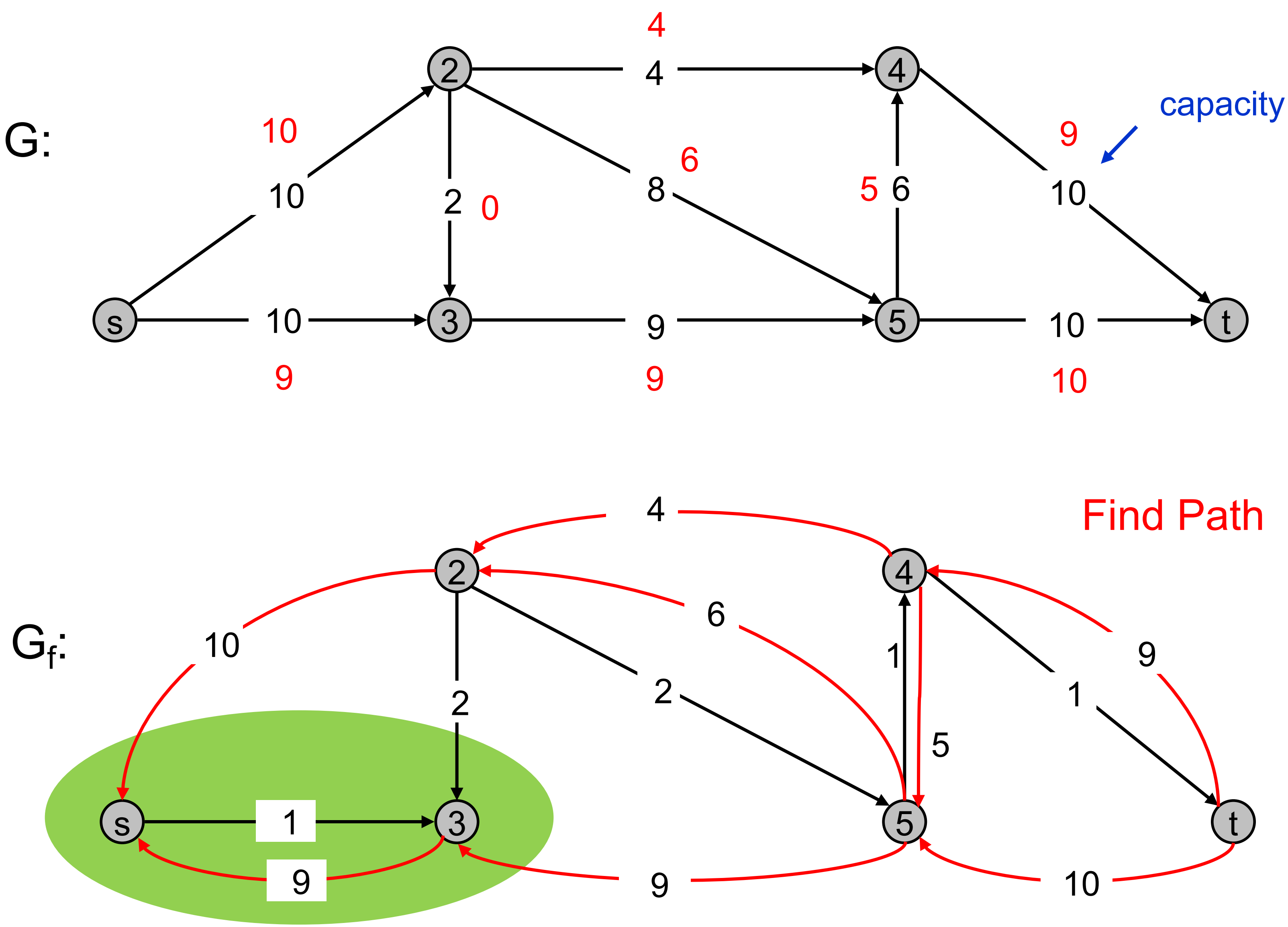


# Augmenting Path Algorithm

```
Augment(f, c, P) {
   b ← bottleneck(P)
   foreach e ∈ P {
      if (e ∈ E) f(e) ← f(e) + b
                 c(e) ← c(e) - b
                 c(eᴿ) ← c(eᴿ) + b

      else       f(eᴿ) ← f(eᴿ) - b
                 c(eᴿ) ← c(eᴿ) + b
                 c(e) ← c(e) - b
   }
   return f
}
```

Smallest capacity edge on P

Forward edge

Reverse edge

```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0. Gf is residual graph
   while (there exists augmenting path P) {
      f ← Augment(f, c, P)
}
   return f
}
```

# Max Flow Min Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Ford-Fulkerson 1956] The value of the max s-t flow is equal to the value of the min s-t cut.

Proof strategy. We prove both simultaneously by showing the TFAE:

(i) There exists a cut (A, B) such that v(f) = cap(A, B).

(ii) Flow f is a max flow.

(iii) There is no augmenting path relative to f.

(i) $\Rightarrow$ (ii) This was the corollary to weak duality lemma.

(ii) $\Rightarrow$ (iii) We show contrapositive.

Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along that path.

# Pf of Max Flow Min Cut Theorem

(iii) => (i)

No augmenting path for f => there is a cut (A,B): v(f)=cap(A,B)

- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s in residual graph.
- By definition of A, s $\in$ A.
- By definition of f, t $\notin$ A.

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
&= \sum_{e \text{ out of } A} c(e) \\
&= cap(A,B)
\end{aligned}
$$

# Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value $f(e)$ and every residual capacities $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \leq nC$ iterations, if $f^*$ is optimal flow.
Pf. Each augmentation increase value by at least 1.

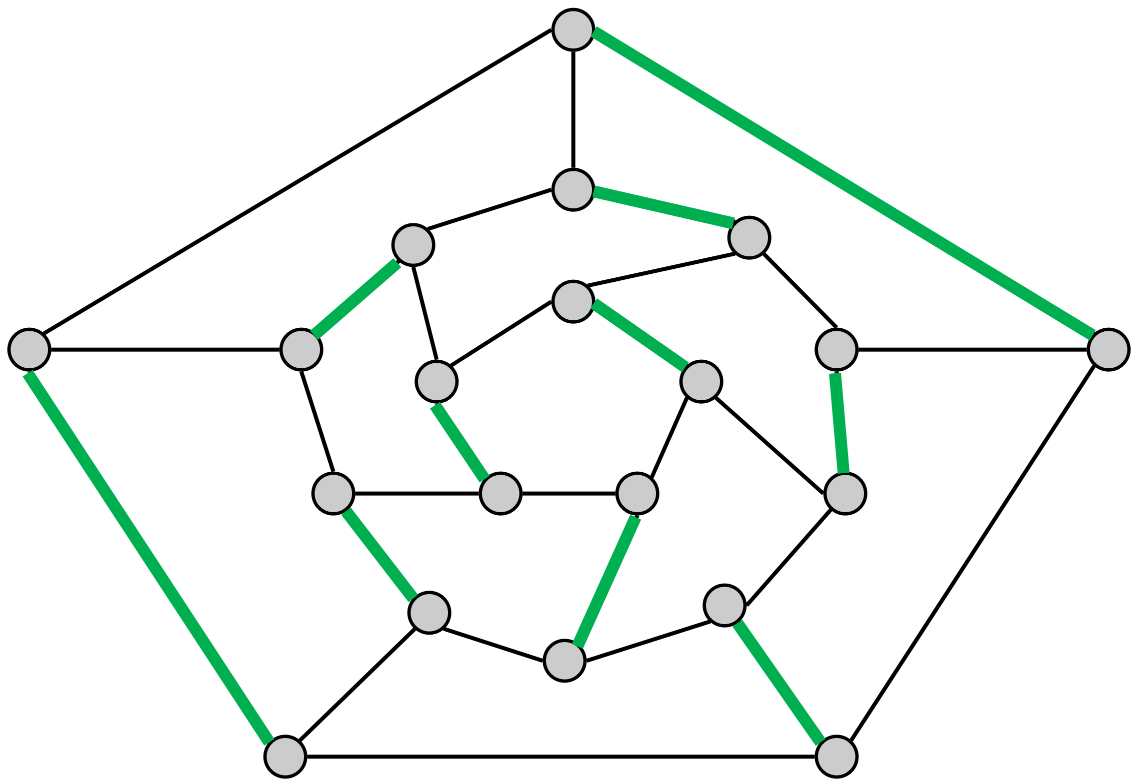
Corollary. If C = 1, Ford-Fulkerson runs in O(mn) time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer.
Pf. Since algorithm terminates, theorem follows from invariant.

# Applications of Max Flow: Bipartite Matching

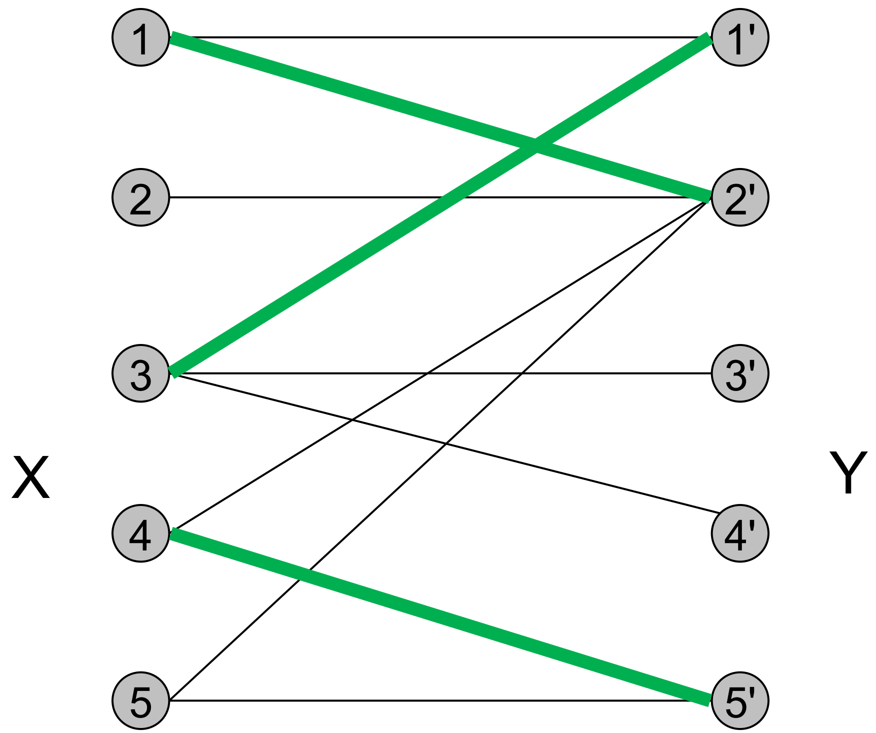
# Maximum Matching Problem

Given an undirected graph G = (V, E).
A set $M \subseteq E$ is a matching if each node appears in at most one edge in M.
Goal: find a matching with largest cardinality.



# Bipartite Matching Problem

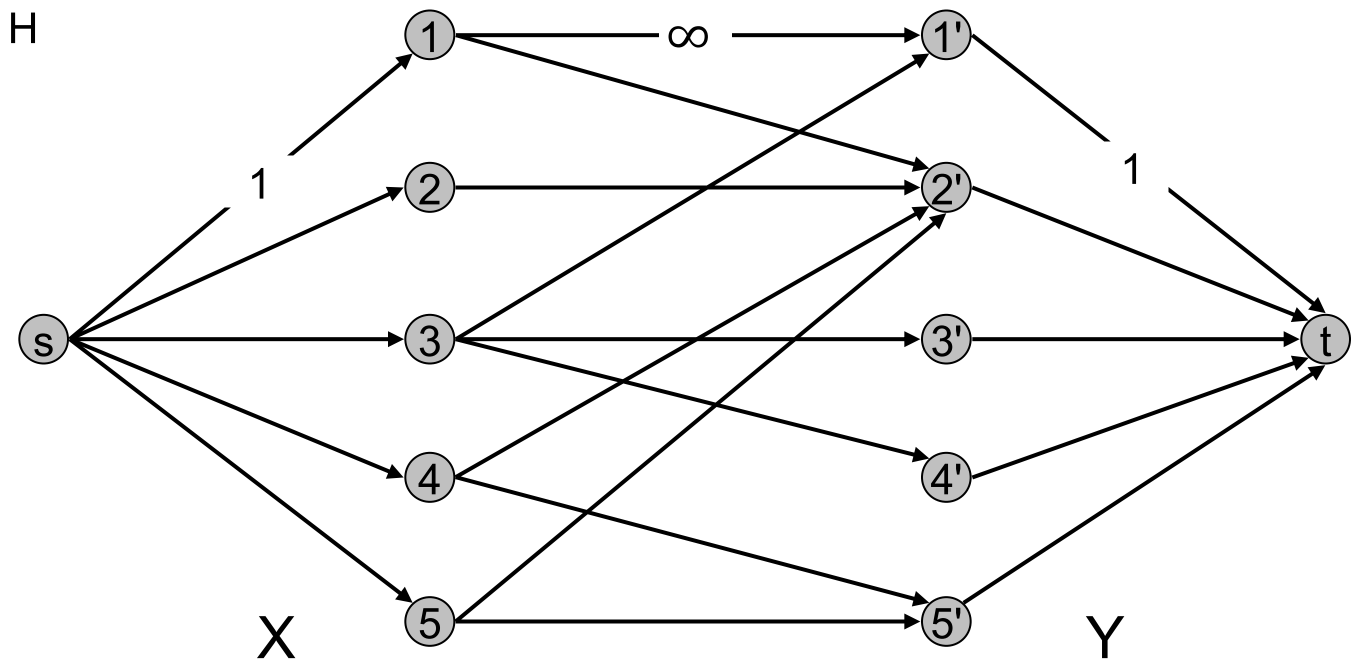
Given an undirected bibpartite graph $G = (X \cup Y, E)$
A set $M \subseteq E$ is a matching if each node appears in at most one edge in M.
Goal: find a matching with largest cardinality.



# Bipartite Matching using Max Flow

Create digraph H as follows:

- Orient all edges from X to Y, and assign infinite (or unit) capacity.
- Add source s, and unit capacity edges from s to each node in L.
- Add sink t, and unit capacity edges from each node in R to t.



# Bipartite Matching: Proof of Correctness

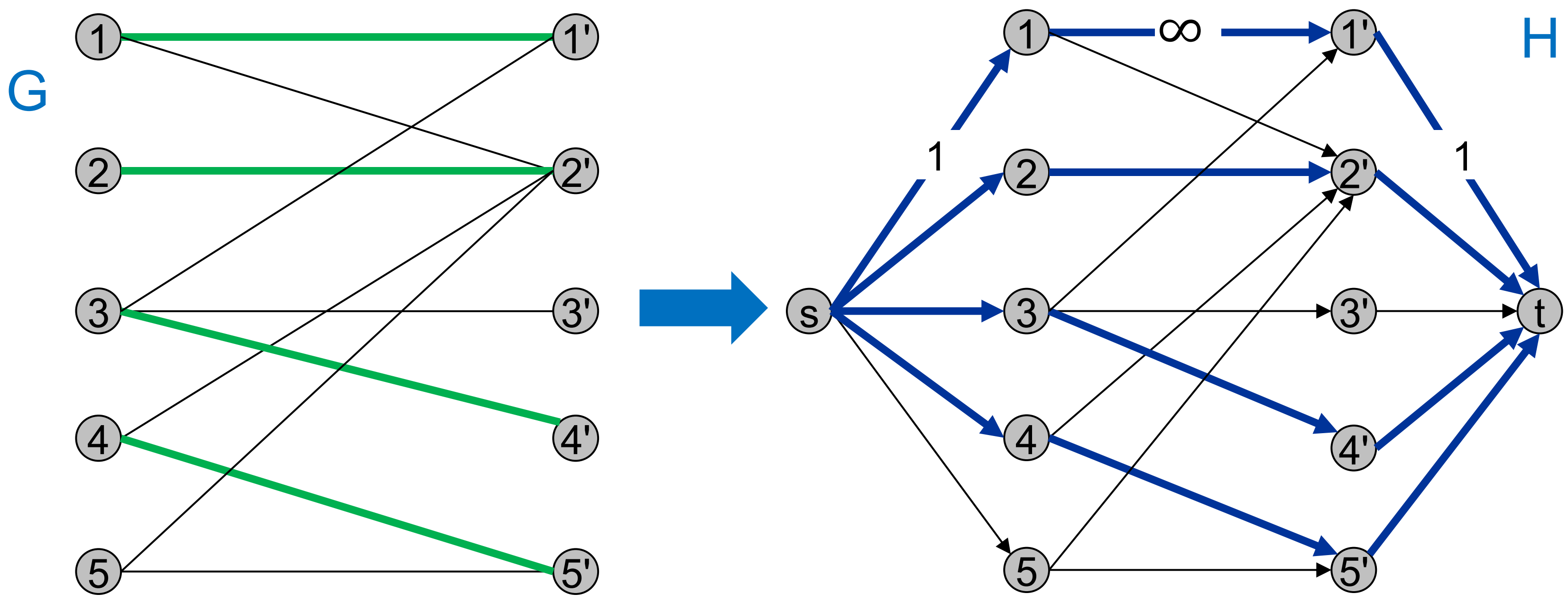
Thm. Max cardinality matching in G = value of max flow in H.

Pf. $\leq$

Given max matching M of cardinality k.

Consider flow f that sends 1 unit along each of k edges of M.

f is a flow, and has cardinality k. ▪



# Bipartite Matching: Proof of Correctness

Thm. Max cardinality matching in G = value of max flow in H.

Pf. (of $\geq$) Let f be a max flow in H of value k.

Integrality theorem $\Rightarrow$ k is integral and we can assume f is 0-1.

Consider M = set of edges from X to Y with f(e) = 1.

- each node in X and Y participates in at most one edge in M
- |M| = k: consider s-t cut $(s \cup X, t \cup Y)$