CSE 421: Introduction to Algorithms

Stable Matching

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Matching Residents to Hospitals

Goal: Given a set of preferences among hospitals and medical school residents (graduating medical students), design a self-reinforcing admissions process.

Unstable pair: applicant **A** and hospital **Y** are unstable if:

A prefers **Y** to its assigned hospital.

Y prefers **A** to one of its admitted applicants.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital side deal from being made.

Simpler: Stable Matching Problem

Given n hetero men $m_1, \ldots, m_n$,
and n hetero women, $w_1, \ldots, w_n$
find a "stable matching".

- Participants rate members of opposite sex.
- Each man lists women in order of preference.
- Each woman lists men in order of preference.

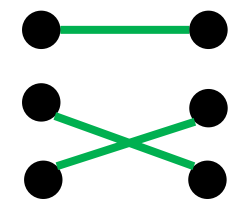
	favorite		least favorite
	1st	2nd	3rd
m_1	w_1	w_2	w_3
m_2	w_2	w_1	w_3
m_3	w_1	w_2	w_3

	favorite		least favorite
	1st	2nd	3rd
w_1	m_2	m_1	m_3
w_2	m_1	m_2	m_3
w_3	m_1	m_2	m_3

Stable Matching

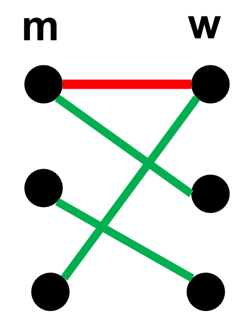
Perfect matching:

- Each man gets exactly one woman.
- Each woman gets exactly one man.



Stability: no incentive for some pair of participants to undermine assignment by joint action.

In a matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.



Stable matching: perfect matching with no unstable pairs.

Stable matching problem: Given the preference lists of n men and n women, find a stable matching if one exists.

Example

Question. Is assignment $(m_1, w_3), (m_2, w_2), (m_3, w_1)$ stable?

	1st (favorite)	2nd	3rd (least favorite)
m_1	w_1	w_2	w_3
m_2	w_2	w_1	w_3
m_3	w_1	w_2	w_3

	1st (favorite)	2nd	3rd (least favorite)
w_1	m_2	m_1	m_3
w_2	m_1	m_2	m_3
w_3	m_1	m_2	m_3

Example

Question. Is assignment (m_1, w_3), (m_2,w_2), (m_3,w_1) stable?

Answer. No. w_2, m_1 will hook up.

	favorite 1st	2nd	least favorite 3rd
m_1	w_1	w_2	w_3
m_2	w_2	w_1	w_3
m_3	w_1	w_2	w_3

	favorite 1st	2nd	least favorite 3rd
w_1	m_2	m_1	m_3
w_2	m_1	m_2	m_3
w_3	m_1	m_2	m_3

Example

Question: Is assignment $(m_1, w_1), (m_2, w_2), (m_3, w_3)$ stable?

Answer: Yes.

	1st (favorite)	2nd	3rd (least favorite)
m_1	w_1	w_2	w_3
m_2	w_2	w_1	w_3
m_3	w_1	w_2	w_3

	1st (favorite)	2nd	3rd (least favorite)
w_1	m_2	m_1	m_3
w_2	m_1	m_2	m_3
w_3	m_1	m_2	m_3

Existence of Stable Matchings

Question. Do stable matchings always exist?
Answer. Yes, but not obvious a priori.

Stable roommate problem:

2n people; each person ranks others from **1** to **2n-1**.
Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
David	A	B	C

A-B, C-D $\Rightarrow$ B-C unstable
A-C, B-D $\Rightarrow$ A-B unstable
A-D, B-C $\Rightarrow$ A-C unstable

So, Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm [Gale-Shapley'62]

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

First step: Properties of Algorithm

Observation 1: Men propose to women in decreasing order of preference.

Observation 2: Each man proposes to each woman at most once

Observation 3: Once a woman is matched, she never becomes unmatched; she only "trades up."

What do we need to prove?

1) The algorithm ends
   - How many steps does it take?

2) The algorithm is correct [usually the harder part]
   - It outputs a perfect matching
   - The output matching is stable

1) Termination

Claim. Algorithm terminates after $\leq \boldsymbol{n^2}$ iterations of while loop.
Proof. Observation 2: Each man proposes to each woman at most once.
Each man makes at most n proposals
So, there are only n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Walter	B	C	D	A	E
Xavier	C	D	A	B	E
Yuri	D	A	B	C	E
Zoran	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Brenda	X	Y	Z	V	W
Claire	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

2) Correctness: Output is Perfect matching

Claim. All men and women get matched.

Proof. (by contradiction)

Suppose, for sake of contradiction, that m_1 is not matched upon termination of algorithm.

Then some woman, say w_1, is not matched upon termination.

By Observation 3 (only trading up, never becoming unmatched), w_1 was never proposed to.

But, m_1 proposes to everyone, since he ends up unmatched.

$\blacksquare$

2) Correctness: Stability

Claim. No unstable pairs.

Proof. (by contradiction)

Suppose m, w is an unstable pair: each prefers each other to the partner in Gale-Shapley matching **S***.

Obs1: men propose in decreasing order of preference

Case 1: m never proposed to w.

$\Rightarrow$ m prefers his **S*** partner to w.

$\Rightarrow$ m, w is stable.

Case 2: m proposed to w.

$\Rightarrow$ w rejected m (right away or later)

$\Rightarrow$ w prefers her **S*** partner to m.

$\Rightarrow$ m, w is stable.

Obs3: women only trade up

In either case m, w is stable, a contradiction. $\blacksquare$

Summary

Stable matching problem: Given n men and n women, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm: Guarantees to find a stable matching for any problem instance.

- Q: How to implement GS algorithm efficiently?

- Q: If there are multiple stable matchings, which one does GS find?

- Q: How many stable matchings are there?