CSE 421

Divide and Conquer: Integer Multiplication

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Closest Pair of Points (2-dimensions)

Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

Brute force: Check all pairs of points p and q with $\Theta(n^2)$ time.

Assumption: No two points have same x coordinate.

A Divide and Conquer Alg

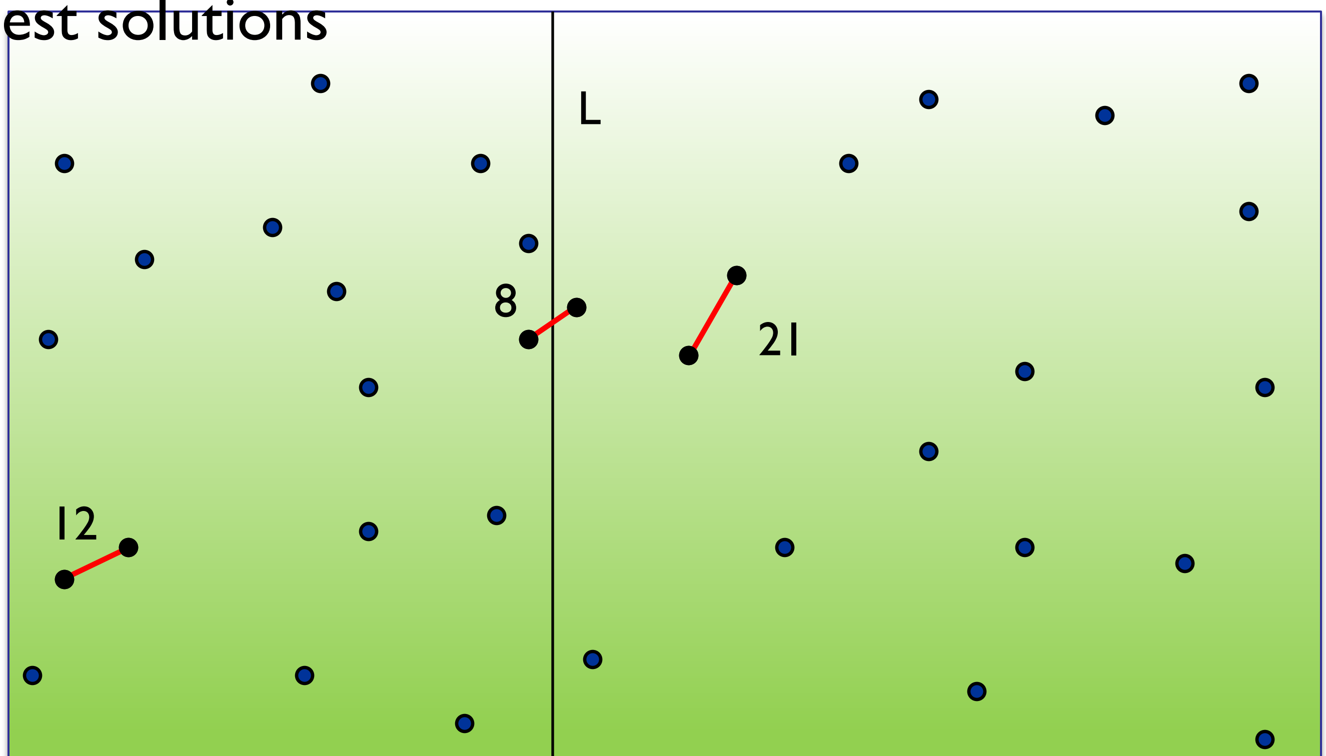
Divide: draw vertical line L with ≈ **n/2 points on each side.**

Conquer: find closest pair on each side, recursively.

Combine to find closest pair overall

seems like $\Theta(n^2)$?

Return best solutions



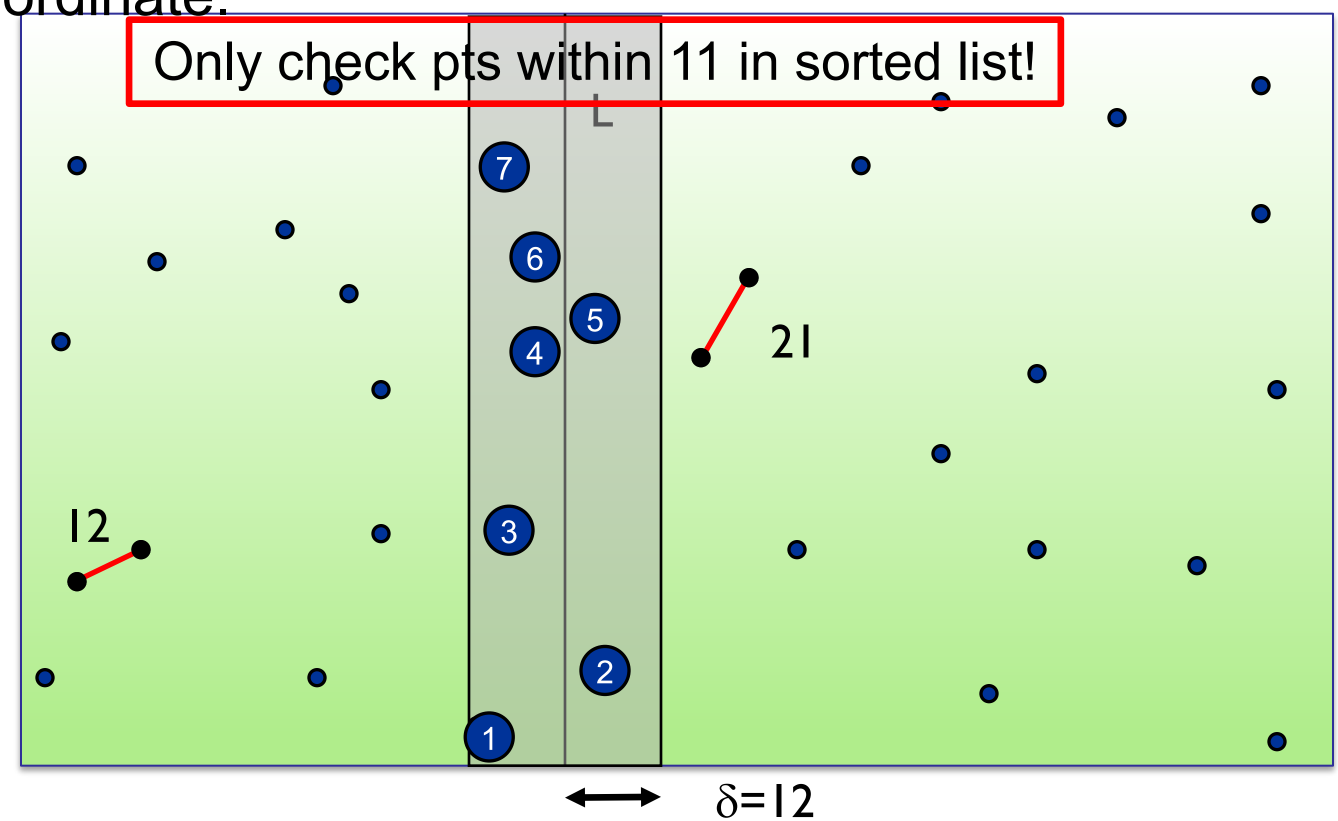
Key Observation

Suppose δ is the minimum distance of all pairs in left/right of L.

$$\delta = \min(12{,}21) = 12.$$

Key Observation: suffices to consider points within δ of line L.

Almost the one-D problem again: Sort points in 2δ-strip by their y coordinate.



Almost 1D Problem

Partition each side of L into $\frac{\delta}{2} \times \frac{\delta}{2}$ squares

Claim: No two points lie in the same $\frac{\delta}{2} \times \frac{\delta}{2}$ box.

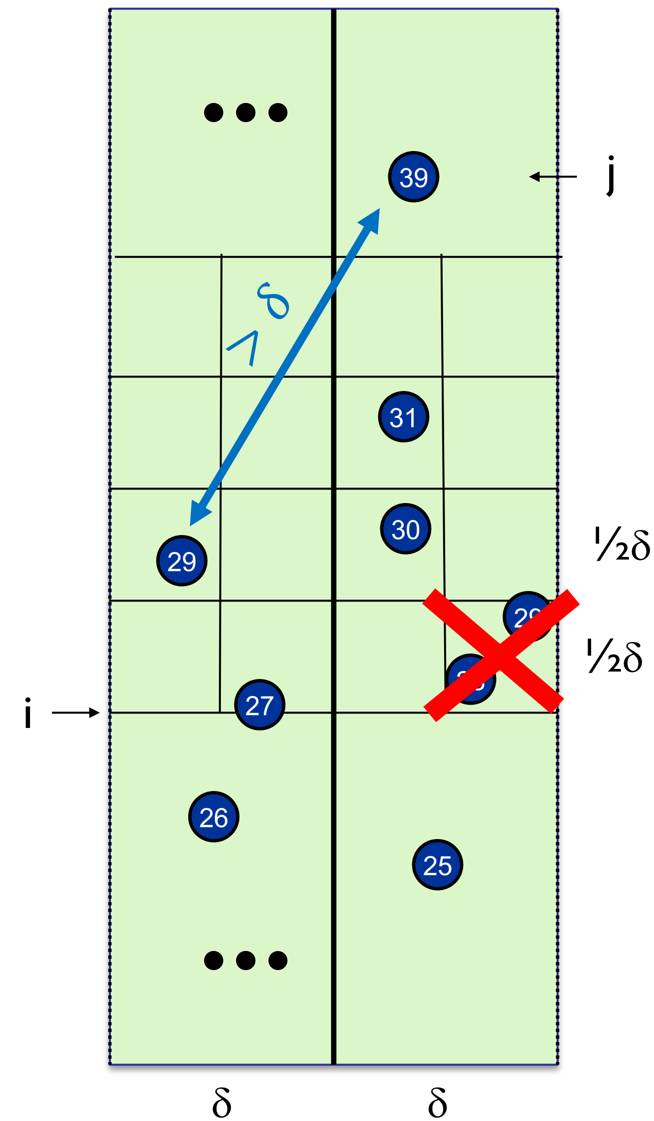
Pf: Such points would be within

$$\sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta}{2}\right)^2} = \delta\sqrt{\frac{1}{2}} \approx 0.7\delta < \delta$$

Let s_i have the i^{th} smallest y-coordinate among points in the 2δ-width-strip.

Claim: If $|i-j| > 11$, then the distance between s_i and s_j is $> \delta$.

Pf: only 11 boxes within δ of $y(s_i)$.



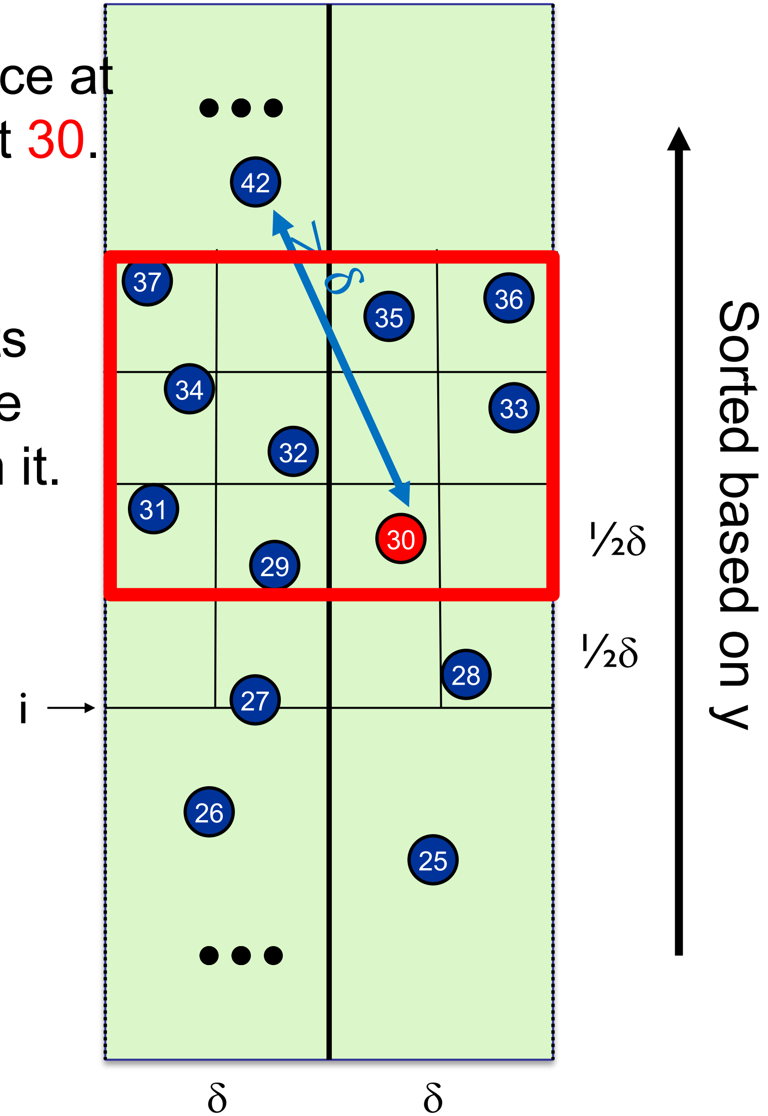
Recap: Finding Closest Pair

Point 42 has distance at least 2δ from point 30.

At most 11 points ahead of 30 have distance $< \delta$ from it.

So, enough to check distance

Distance of 30 to 19…41.



Closest Pair (2Dim Algorithm)

```
Closest-Pair(p_1, …, p_n) {
   if(n <= ??) return ??

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ_1 = Closest-Pair(left half)
   δ_2 = Closest-Pair(right half)
   δ  = min(δ_1, δ_2)

   Delete all points further than δ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      for k = 1…11
        if i+k <= m
            δ = min(δ, distance(p[i], p[i+k]));

   return δ.
}
```

Closest Pair Analysis I

Let D(n) be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on n ≥ 1 points

$$D(n) \leq \begin{cases} 1 & \text{if } n = 1 \\ 2D\left(\frac{n}{2}\right) + 11\,n & \text{o.w.} \end{cases} \Rightarrow D(n) = \mathrm{O}(n\log n)$$

BUT, that's only the number of *distance calculations*

What if we counted running time?

$$T(n) \leq \begin{cases} 1 & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + O(n\log n) & \text{o.w.} \end{cases} \Rightarrow D(n) = \mathrm{O}(n\log^2 n)$$

Can we do better? (Analysis II)

Yes!!

Don’t sort by y-coordinates each time.

Sort by x at top level only.

This is enough to divide into two equal subproblems in O(n)

Each recursive call returns δ and list of all points sorted by y

Sort points by y-coordinate by merging two pre-sorted lists.

$$T(n) \leq \begin{cases} 1 & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + O(n) & \text{o.w.} \end{cases} \Rightarrow D(n) = O(n \log n)$$

Master Theorem

Suppose $T(n) = a\,T\left(\frac{n}{b}\right) + cn^k$ for all $n > b$. Then,

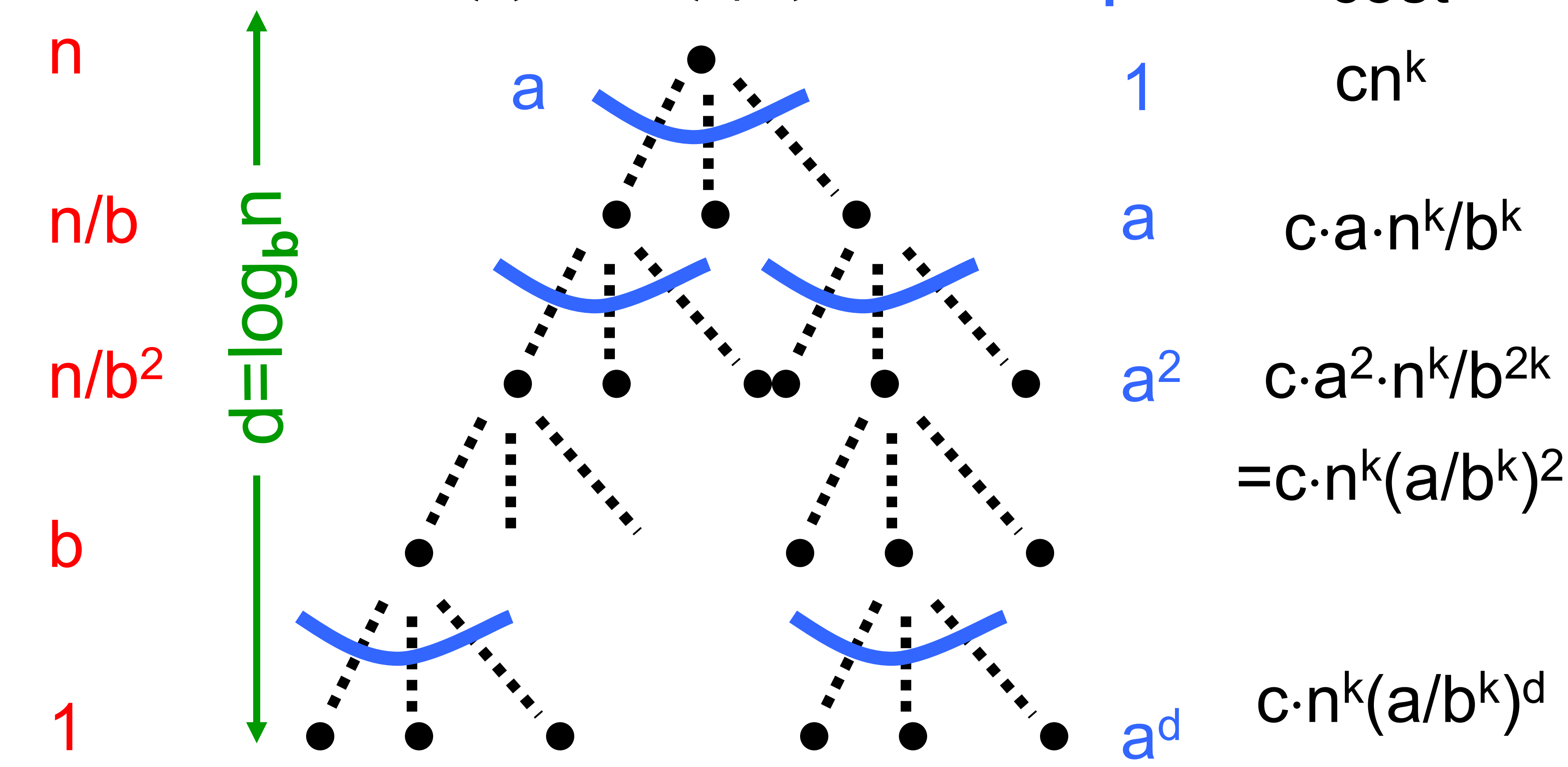
- If $a > b^k$ then $T(n) = \Theta\left(n^{\log_b a}\right)$
- If $a < b^k$ then $T(n) = \Theta\left(n^k\right)$
- If $a = b^k$ then $T(n) = \Theta\left(n^k \log n\right)$

Works even if it is $\left\lceil\frac{n}{b}\right\rceil$ instead of $\frac{n}{b}$.

We also need $a \geq 1, b > 1\,, k \geq 0$ and $T(n) = O(1)$ for $n \leq b$.

Proving Master Theorem

Problem size $T(n) = aT(n/b) + cn^k$ # probs cost



$$T(n) = cn^k \sum_{i=0}^{d=\log_b n} \left(\frac{a}{b^k}\right)^i$$

A Useful Identity

Theorem: $1 + x + x^2 + \cdots + x^d = \frac{x^{d+1}-1}{x-1}$

Pf: Let $S = 1 + x + x^2 + \cdots + x^d$

Then, $xS = x + x^2 + \cdots + x^{d+1}$

So, $xS - S = x^{d+1} - 1$
i.e., $S(x-1) = x^{d+1} - 1$
Therefore,

$$S = \frac{x^{d+1} - 1}{x - 1}$$

Solve: $T(n) = aT\left(\frac{n}{b}\right) + cn^k,\ a > b^k$

$$T(n) = cn^k \sum_{i=0}^{\log_b n} \left(\frac{a}{b^k}\right)^i$$

$\frac{x^{d+1}-1}{x-1}$ for $x = \frac{a}{b^k}$

$d = \log_b n$

using $x \neq 1$

$$= cn^k \frac{\left(\frac{a}{b^k}\right)^{\log_b n+1} - 1}{\left(\frac{a}{b^k}\right) - 1}$$

$b^{k \log_b n}$

$= \left(b^{\log_b n}\right)^k$

$= n^k$

$$= c\left(\frac{n^k}{b^{k \log_b n}}\right) \frac{\left(\frac{a}{b^k}\right)}{\left(\frac{a}{b^k}\right) - 1}\, a^{\log_b n}$$

$a^{\log_b n}$

$= \left(b^{\log_b a}\right)^{\log_b n}$

$= \left(b^{\log_b n}\right)^{\log_b a}$

$= n^{\log_b a}$

$$\leq c'\, a^{\log_b n} = O\left(n^{\log_b a}\right)$$

Solve: $T(n) = aT\left(\frac{n}{b}\right) + cn^k$, $a = b^k$

$$T(n) = cn^k \sum_{i=0}^{\log_b n} \left(\frac{a}{b^k}\right)^i$$

$$= cn^k \log_b n$$

Master Theorem

Suppose $T(n) = a\,T\left(\frac{n}{b}\right) + cn^k$ for all $n > b$. Then,

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Works even if it is $\left\lceil\frac{n}{b}\right\rceil$ instead of $\frac{n}{b}$.

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Integer Multiplication

Integer Arithmetic

Add: Given two n-bit integers a and b, compute a + b.

Add

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

O(n) bit operations.

Multiply: Given two n-bit integers a and b, compute a × b. The "grade school" method:

Multiply

								1	1	0	1	0	1	0	1
							*	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1		
					1	1	0	1	0	1	0	1			
				1	1	0	1	0	1	0	1				
			1	1	0	1	0	1	0	1					
		1	1	0	1	0	1	0	1						
	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

$O(n^2)$ bit operations.

How to use Divide and Conquer?

Suppose we want to multiply two 2-digit integers (32,45).

We can do this by multiplying four 1-digit integers

Then, use add/shift to obtain the result:

		4	5	y_1 y_0
		3	2	x_1 x_0
		1	0	$x_0 \cdot y_0$
	0	8		$x_0 \cdot y_1$
	1	5		$x_1 \cdot y_0$
1	2			$x_1 \cdot y_1$
1	4	4	0	

$$
\begin{aligned}
x &= 10x_1 + x_0 \\
y &= 10y_1 + y_0 \\
xy &= (10x_1 + x_0)(10y_1 + y_0) \\
&= 100\, x_1y_1 + 10(x_1y_0 + x_0y_1) + x_0y_0
\end{aligned}
$$

Same idea works when multiplying n-digit integers:

- Divide into 4 n/2-digit integers.
- Recursively multiply
- Then merge solutions

A Divide and Conquer for Integer Mult

Let x, y be two n-bit integers

Write $x = 2^{n/2}x_1 + x_0$ and $y = 2^{n/2}y_1 + y_0$

where x_0, x_1, y_0, y_1 are all n/2-bit integers.

$$\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0
\end{aligned}$$

Therefore,

$$T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n)$$

We only need 3 values
$x_1 y_1, x_0 y_0, x_1 y_0 + x_0 y_1$
Can we find all 3 by only 3 multiplication?

So,

$$T(n) = \Theta(n^2).$$

Key Trick: 4 multiplies at the price of 3

$$x = 2^{n/2} \cdot x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0$$

$$\alpha = x_1 + x_0$$
$$\beta = y_1 + y_0$$
$$\alpha\beta = (x_1 + x_0)(y_1 + y_0)$$
$$= x_1 y_1 + (x_1 y_0 + x_0 y_1) + x_0 y_0$$
$$(x_1 y_0 + x_0 y_1) = \alpha\beta - x_1 y_1 - x_0 y_0$$

Key Trick: 4 multiplies at the price of 3

Theorem [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in O($n^{1.585\ldots}$) bit operations.

$$x = 2^{n/2} \cdot x_1 + x_0 \Rightarrow \alpha = x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0 \Rightarrow \beta = y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= 2^n \cdot \underbrace{x_1 y_1}_{A} + 2^{n/2} \cdot \underbrace{(x_1 y_0 + x_0 y_1)}_{\alpha\beta - A - B} + \underbrace{x_0 y_0}_{B}$$

To multiply two n-bit integers:

Add two n/2 bit integers.

Multiply three n/2-bit integers.

Add, subtract, and shift n/2-bit integers to obtain result.

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n) \Rightarrow T(n) = O\left(n^{\log_2 3}\right) = O(n^{1.585\ldots})$$

Integer Multiplication (Summary)

- Naïve: $\Theta(n^2)$
- Karatsuba: $\Theta(n^{1.585\ldots})$
- Amusing exercise: generalize Karatsuba to do 5 size n/3 subproblems

  This gives $\Theta(n^{1.46\ldots})$ time algorithm

- Best known algorithm runs in $\Theta(n \log n)$ using fast Fourier transform

  but mostly unused in practice (unless you need really big numbers - a billion digits of π, say)

- Best lower bound $O(n)$: A fundamental open problem

Median

Selecting k-th smallest

Problem: Given numbers $x_1, \ldots, x_n$ and an integer $1 \leq k \leq n$ output the k-th smallest number

$$\mathrm{Sel}(\{x_1, \ldots, x_n\}, k)$$

A simple algorithm: Sort the numbers in time O(n log n) then return the k-th smallest in the array.

Can we do better?

Yes, in time $O(n)$ if $k = 1$ or $k = 2$.

Can we do $O(n)$ for all possible values of k?

Assume all numbers are distinct for simplicity.

An Idea

Choose a number w from $x_1, \dots, x_n$

Define

- $S_<(w) = \{x_i : x_i < w\}$
- $S_=(w) = \{x_i : x_i = w\}$
- $S_>(w) = \{x_i : x_i > w\}$

Can be computed in linear time

Solve the problem recursively as follows:

- If $k \leq |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \leq |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - |S_<(w)| - |S_=(w)|)$

Ideally want $|S_<(w)|, |S_>(w)| \leq n/2$. In this case ALG runs in $O(n) + O\left(\frac{n}{2}\right) + O\left(\frac{n}{4}\right) + \cdots + O(1) = O(n)$.

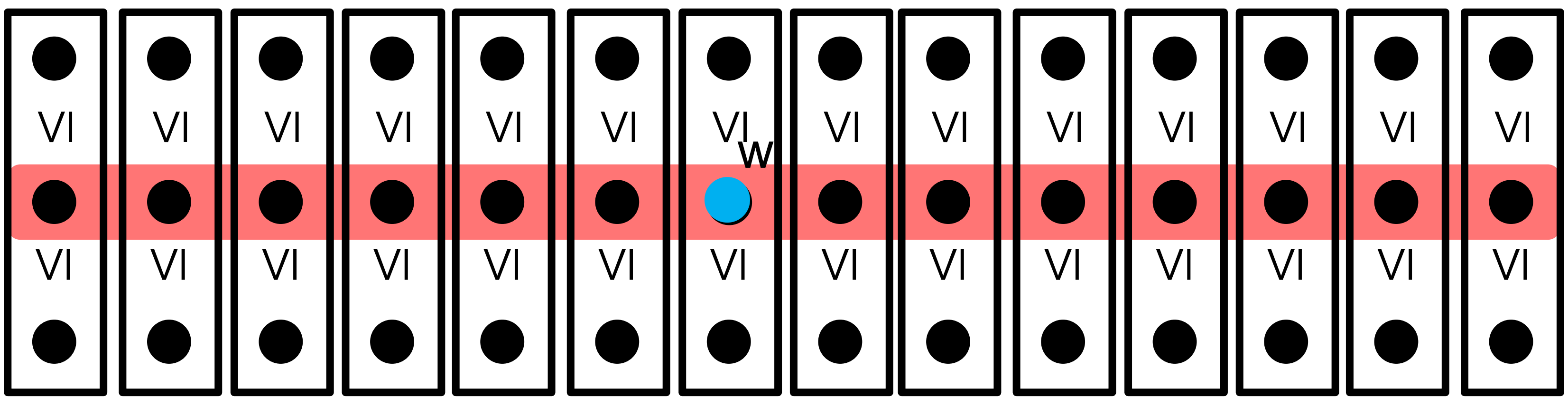
How to choose w?

Suppose we choose w uniformly at random similar to the pivot in quicksort.

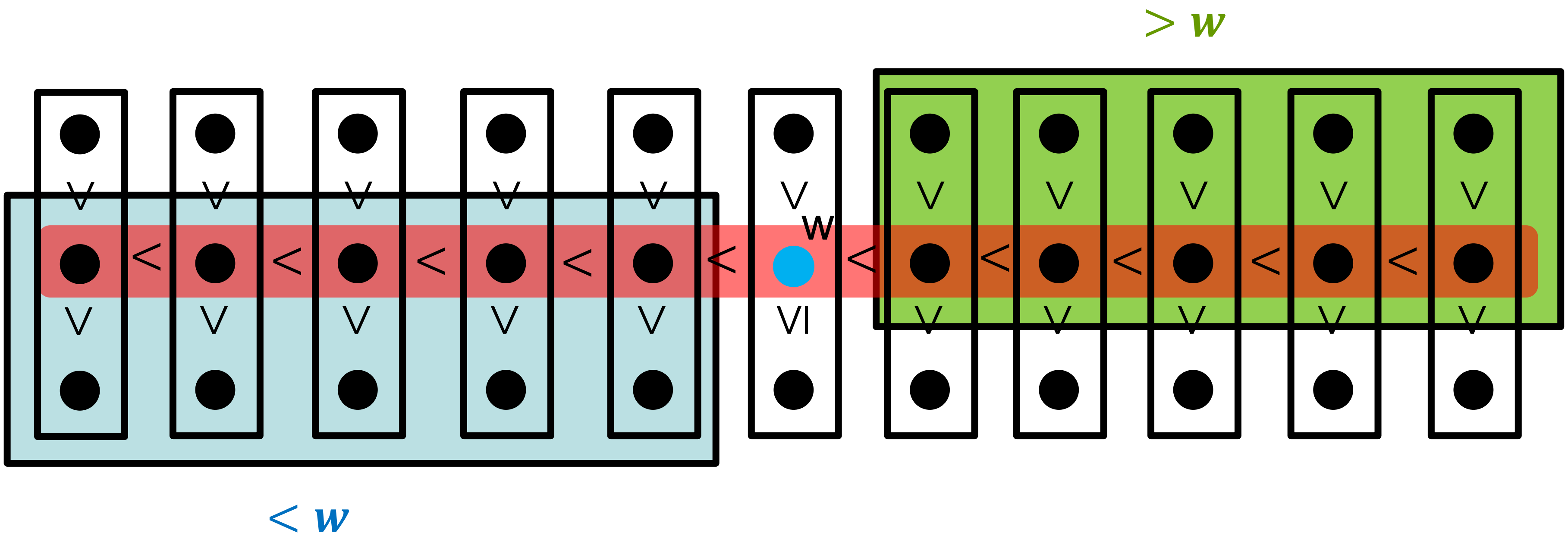
Then, $\mathbb{E}[|S_<(w)|] = \mathbb{E}[|S_>(w)|] = n/2$. Algorithm runs in $O(n)$ in expectation.

Can we get $O(n)$ running time deterministically?

- Partition numbers into sets of size 3.
- Sort each set (takes O(n))
- $w = Sel(midpoints, n/6)$



How to lower bound $|S_<(w)|, |S_>(w)|$?

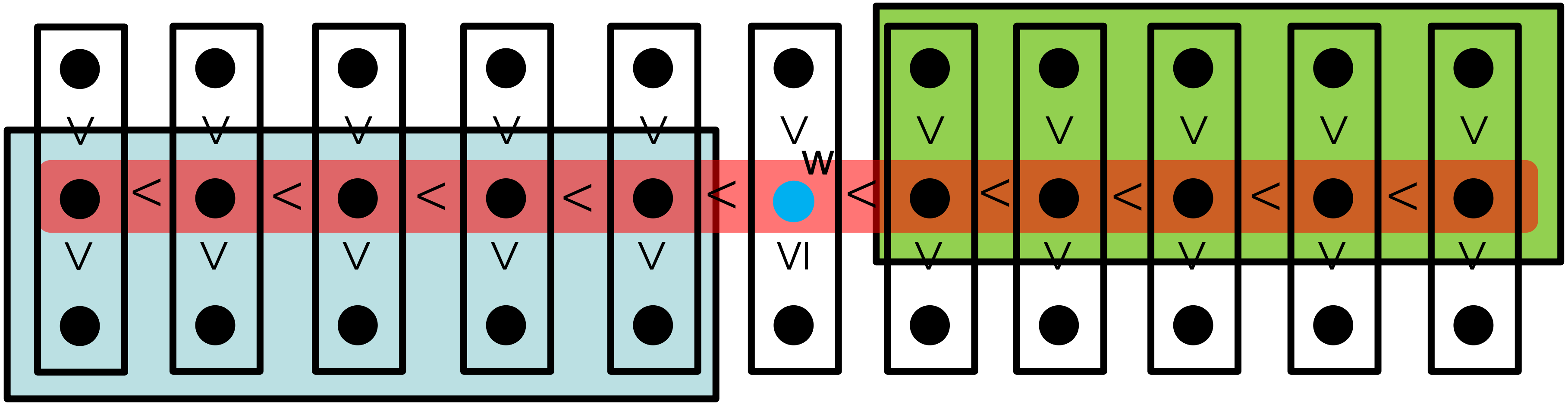


- $|S_<(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$
- $|S_>(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$.

$$\frac{n}{3} \leq |S_<(w)|, |S_>(w)| \leq \frac{2n}{3}$$

So, what is the running time?

Asymptotic Running Time?



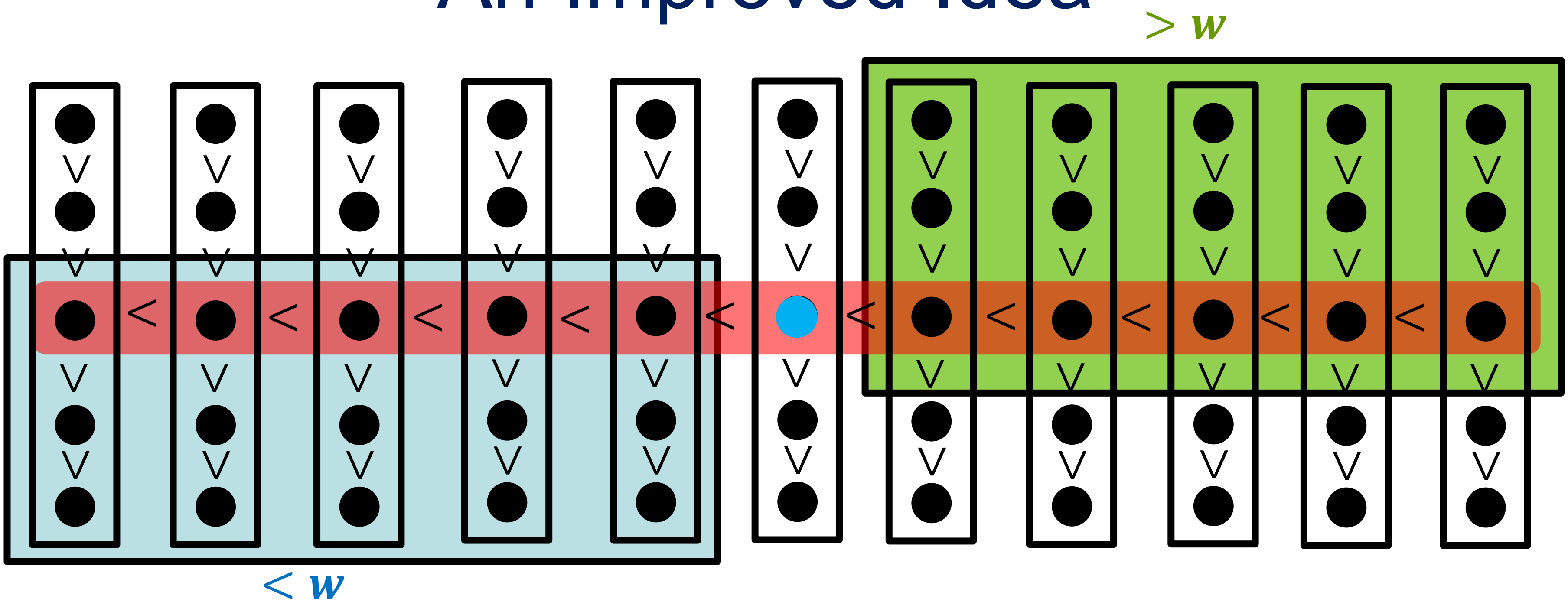
- If $k \le |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \le |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - S_<(w) - S_=(w))$

O(nlog n) again?
So, what is the point?

Where $\frac{n}{3} \le |S_<(w)|, |S_>(w)| \le \frac{2n}{3}$

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + O(n) \Rightarrow T(n) = O(n \log n)$$

An Improved Idea



Partition into n/5 sets. Sort each set and set $w = Sel(midpoints, n/10)$

- $|S_<(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$
- $|S_>(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$

$$\Rightarrow \frac{3n}{10} \leq |S_<(w)|, |S_>(w)| \leq \frac{7n}{10}$$

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n) \Rightarrow T(n) = O(n)$$

An Improved Idea

```
Sel(S, k) {
   n ← |S|
   If (n < ??) return ??
   Partition S into n/5 sets of size 5
   Sort each set of size 5 and let M be the set of medians, so
|M|=n/5
   Let w=Sel(M,n/10)
   For i=1 to n{
     If x_i < w add x to S_<(w)
     If x_i > w add x to S_>(w)
     If x_i = w add x to S_=(w)
   }
   If (k ≤ |S_<(w)|)
     return Sel(S_<(w), k)
   else if (k ≤ |S_<(w)| + |S_=(w)|)
     return w;
   else
     return Sel(S_>(w), k − |S_<(w)| − |S_=(w)|)
}
```

We can maintain each set in an array

D&C Summary

Idea:

"Two halves are better than a whole"

- if the base algorithm has super-linear complexity.

"If a little's good, then more's better"

- repeat above, recursively

- Applications: Many.
  - Binary Search, Merge Sort, (Quicksort),
  - Root of a Function
  - Closest points,
  - Integer multiplication
  - Median
  - Matrix Multiplication