CSE 421

Greedy Alg: Minimum Spanning Tree

Shayan Oveis Gharan

An Advice on Problem Solving

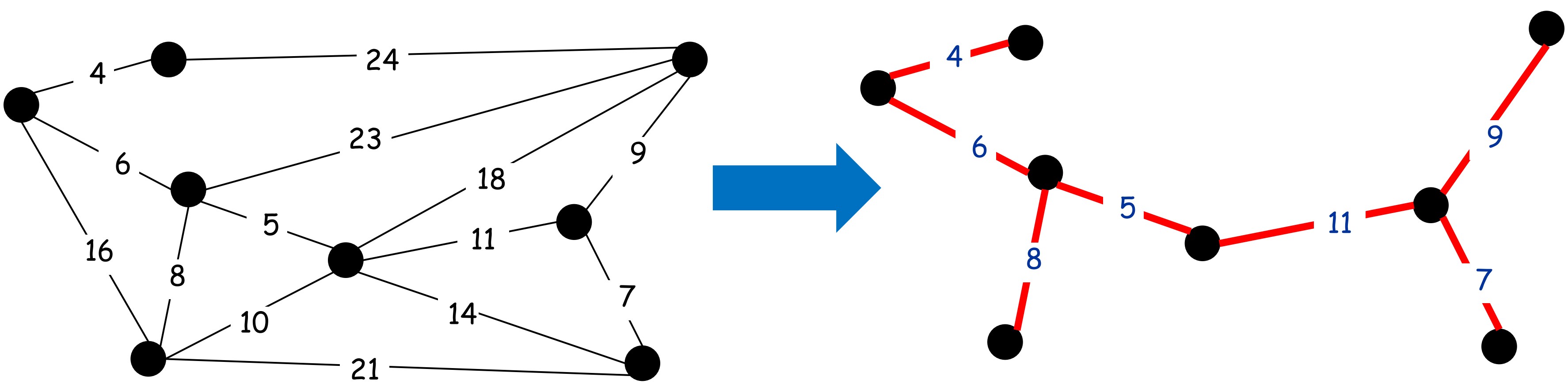
If possible, try not to use arguments of the following type in proofs:

- The Best case is ….
- The worst case is ….
- The slowest running time for my algorithm is ….

These arguments need rigorous justification, and they are usually the main reason that your proofs can become wrong, or unjustified.

Minimum Spanning Tree (MST)

Given a connected graph $G = (V, E)$ with real-valued edge weights c_e, an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge weights is minimized.



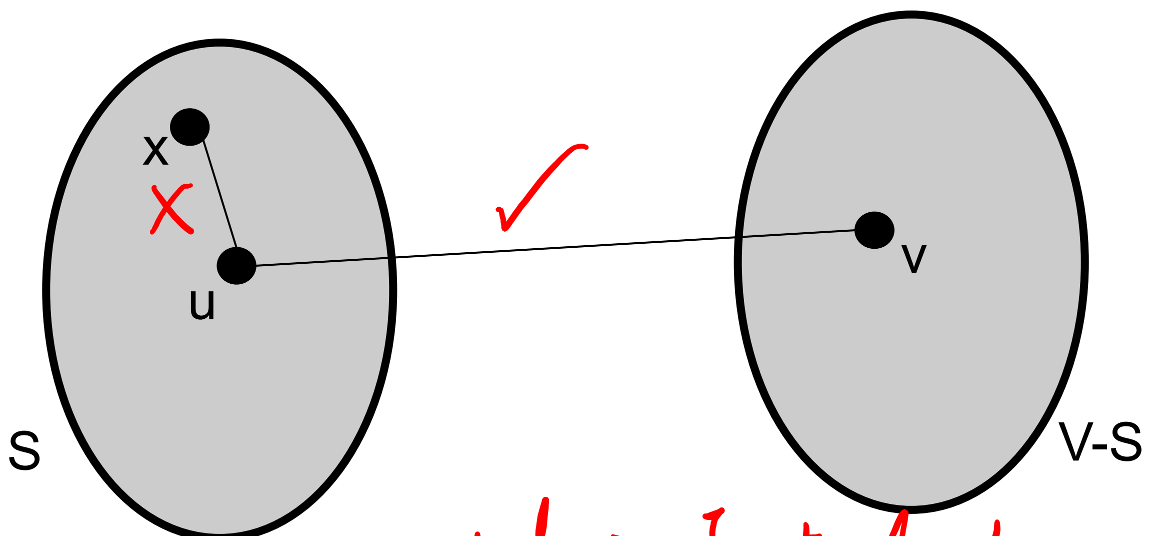
$G = (V, E)$

$$c(T) = \sum_{e \in T} c_e = 50$$

Cuts

In a graph $G = (V, E)$ a cut is a bipartition of V into sets $S, V - S$ for some $S \subseteq V$. We show it by $(S, V - S)$

An edge $e = \{u, v\}$ is in the cut $(S, V - S)$ if exactly one of u,v is in S.

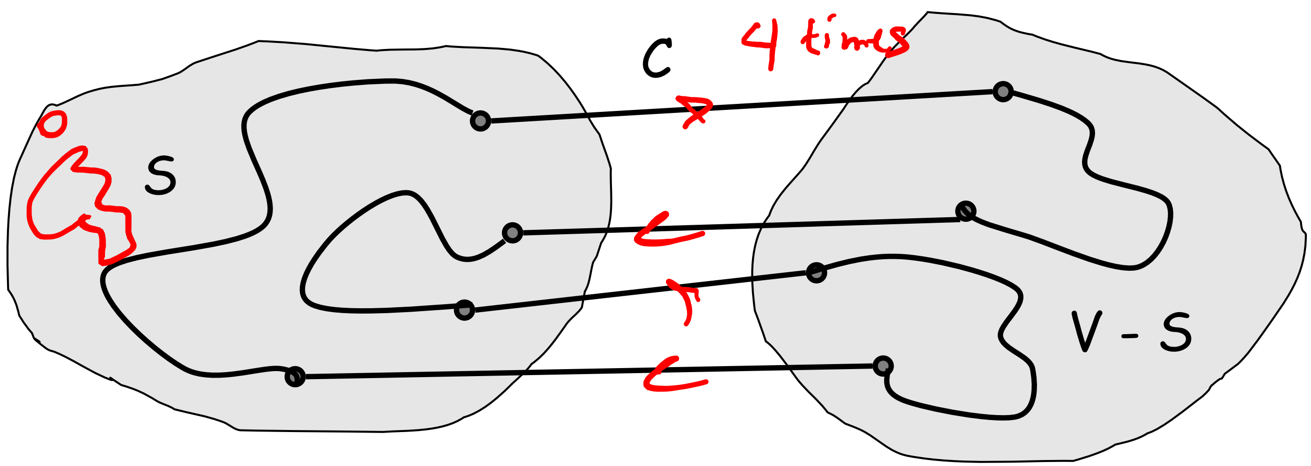


OBS: If G is connected $\Rightarrow$ $\exists$ at least one edge in every cut.

Cycles and Cuts

Claim. A cycle crosses a cut (from S to V-S) an even number of times.

Pf. (by picture)

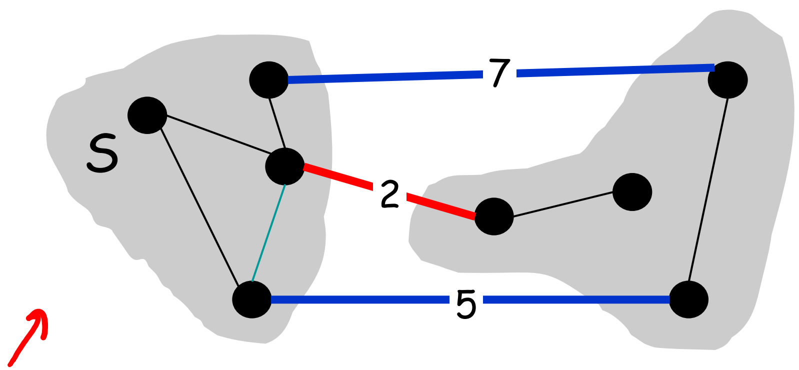


Properties of the OPT

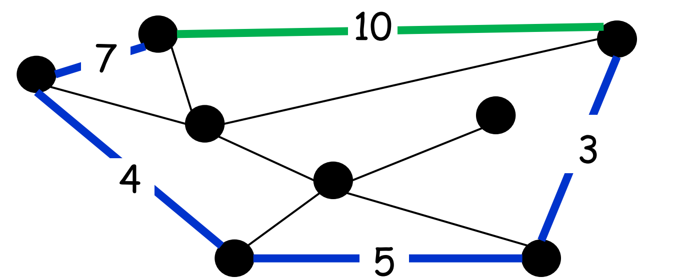
Simplifying assumption: All edge costs c_e are distinct.

Cut property: Let S be any subset of nodes (called a cut), and let e be the min cost edge with exactly one endpoint in S. Then every MST contains e.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then no MST contains f.



red edge is in the MST



Green edge is not in the MST

Cut Property: Proof

Simplifying assumption: All edge costs c_e are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the T* contains e.

Pf. By contradiction

Suppose e = {u,v} does not belong to T*.

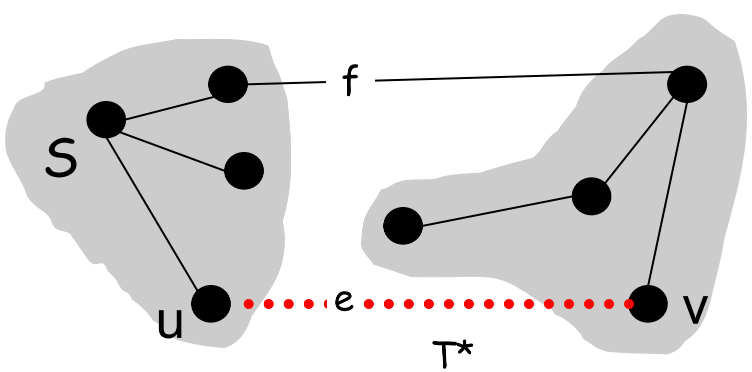
Adding e to T* creates a cycle C in T*.

C crosses S even number of times $\Rightarrow$ there exists another edge, say f, that leaves S.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $c(T) < c(T^*)$.

This is a contradiction.



Cycle Property: Proof

Simplifying assumption: All edge costs c_e are distinct.

Cycle property: Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

Pf. (By contradiction)

Suppose f belongs to T*.

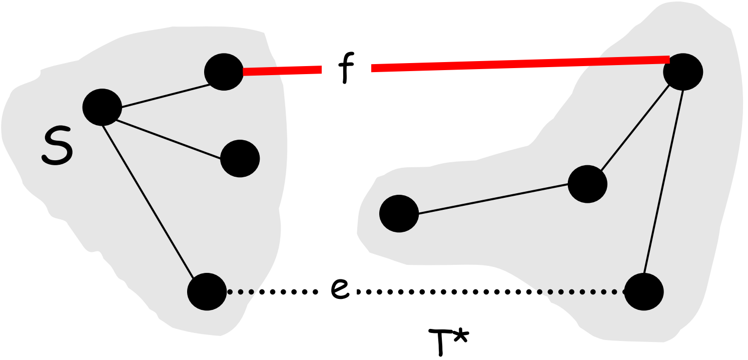
Deleting f from T* cuts T* into two connected components.

There exists another edge, say e, that is in the cycle and connects the components.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $c(T) < c(T^*)$.

This is a contradiction.

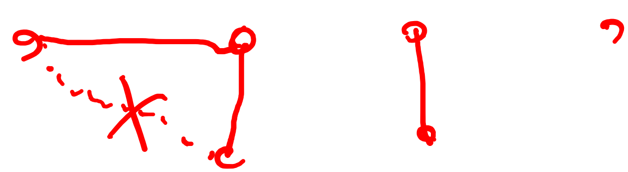


Kruskal's Algorithm [1956]

```
Kruskal(G, c) {
   Sort edges weights so that c1 ≤ c2 ≤ ... ≤ cm.
   T ← ∅

   foreach (u ∈ V) make a set containing singleton {u}

   for i = 1 to m
      Let (u,v) = ei
      if (u and v are in different sets) {
         T ← T ∪ {ei}
         merge the sets containing u and v
      }
   return T
}
```

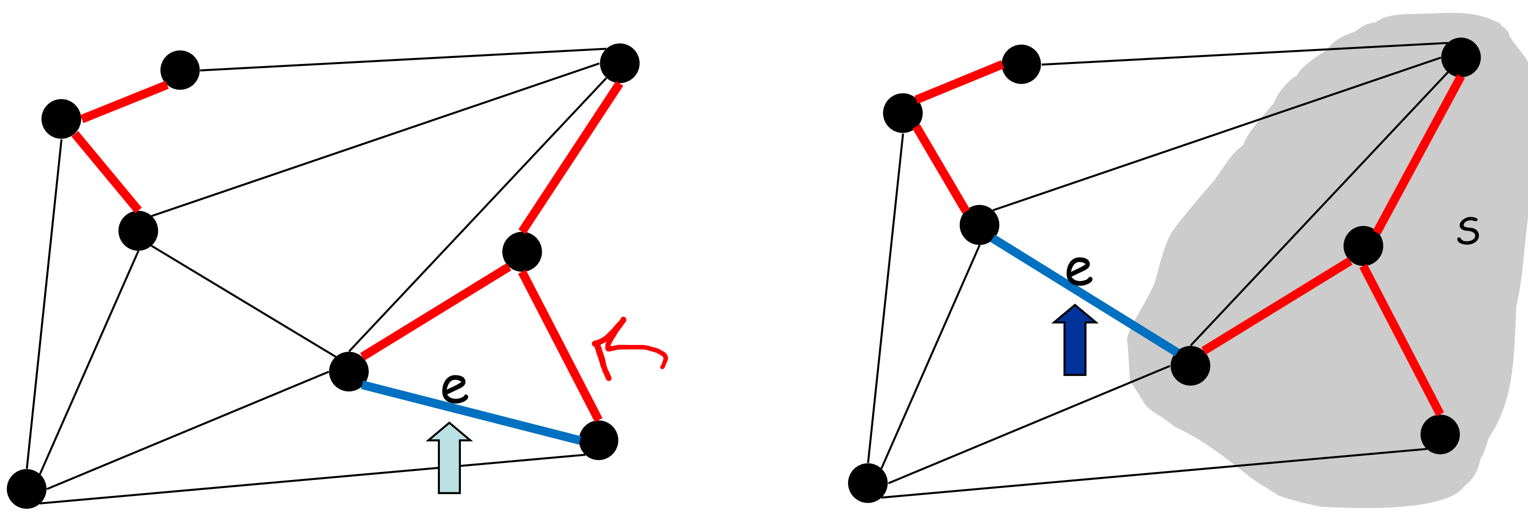


Kruskal's Algorithm: Pf of Correctness

Consider edges in ascending order of weight.

Case 1: If adding e to T creates a cycle, discard e according to cycle property.

Case 2: Otherwise, insert e = (u, v) into T according to cut property where S = set of nodes in u's connected component.



Case 1

Case 2

Implementation: Kruskal's Algorithm

Implementation. Use the union-find data structure.

- Build set T of edges in the MST.
- Maintain a set for each connected component.
- O(m log n) for sorting and O(m log n) for union-find

```
Kruskal(G, c) {
   Sort edges weights so that c₁ ≤ c₂ ≤ ... ≤ cₘ.
   T ← ∅

   foreach (u ∈ V) make a set containing singleton {u}

   for i = 1 to m
      Let (u,v) = eᵢ
      if (u and v are in different sets) {          O(log n)
         T ← T ∪ {eᵢ}
         merge the sets containing u and v
      }                                              O(1)
   return T
}
```