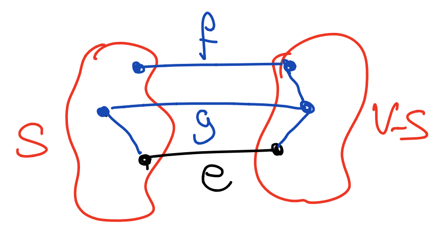
Lem (Cut Property): For any cut $(S, V-S)$, the minimum cost edge e is $(S, V-S)$ is in the MST T^*.

Pf: (By contradiction!)



wrong

T^* is connected, T^* has an edge f in $(S, V-S)$. $c_f > c_e$. Swap f with e. $(T^* - f + e)$ we get a smaller tree. Contradiction!

You need to show $T^* - f + e$ is tree.

Correct Pf: Add e to T^*: $T = T^* \cup \{e\}$.

T has a cycle C_1 and we must have $e \in C_1$

C_1 crosses $(S, V-S)$ even number of times. (one for e) $\Rightarrow$

$\exists$ edge $g \in C_1$ s.t. $g \in (S, V-S)$. Now we claim $T^* - g + e$ is a tree. B/C $c(T^* - g + e) < c(T^*)$ contradiction with T^* MST.

$T^* - g + e$ has $n-1$ edges.

$T^* - g + e$ connected B/C by removing an edge of a cycle you don't disconnect the graph

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Technique: Swapping technique.

Lem (Cycle prop): Let e be the longest edge of a cycle C_1. Then e is not in MST T^*.

Pf: (By contradiction).

Suppose $e \in T^*$. $T = T^* - e \Rightarrow T$ is not a tree so it is disconnected

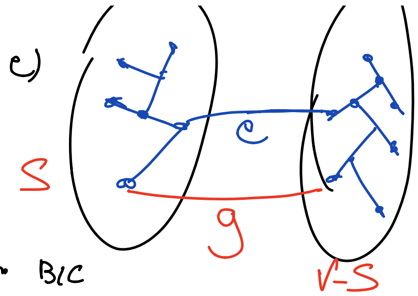
$\Rightarrow \exists (S, V-S)$ that T has no edge.

So e is the only edge of T^* in $(S, V-S)$.

C crosses $(S, V-S)$ even # times (one is e)

$\Rightarrow \exists\, g \in C$ s.t. $g \in (S, V-S)$.

$g \notin T^*$ (e is only edge of T^* in $(S, V-S)$.



Claim $\underline{T^* - e + g}$ is a tree, gives a contradiction B/C

$c(T^* - e + g) < c(T^*)$ since $c(g) < c(e)$.

$T^* - e + g$ has $n-1$ edges.

$T^* - e + g$ has no cycles B/C $T^* - e$ has no cycle and g is the only edge in $(S, V-S)$