CSE 421
Algorithms

Lecture 22

Network Flow, Part 2

Network Flow

Outline

- ~~Network flow definitions~~
- ~~Flow examples~~
- ~~Augmenting Paths~~
- ~~Residual Graph~~
- ~~Ford Fulkerson Algorithm~~
- Cuts
- Maxflow-MinCut Theorem
- Simple applications of Max Flow

Ford-Fulkerson Algorithm (1956)

while not done

Construct residual graph G_R

Find an s-t path P in G_R with capacity $b > 0$

Add b units along in G

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations

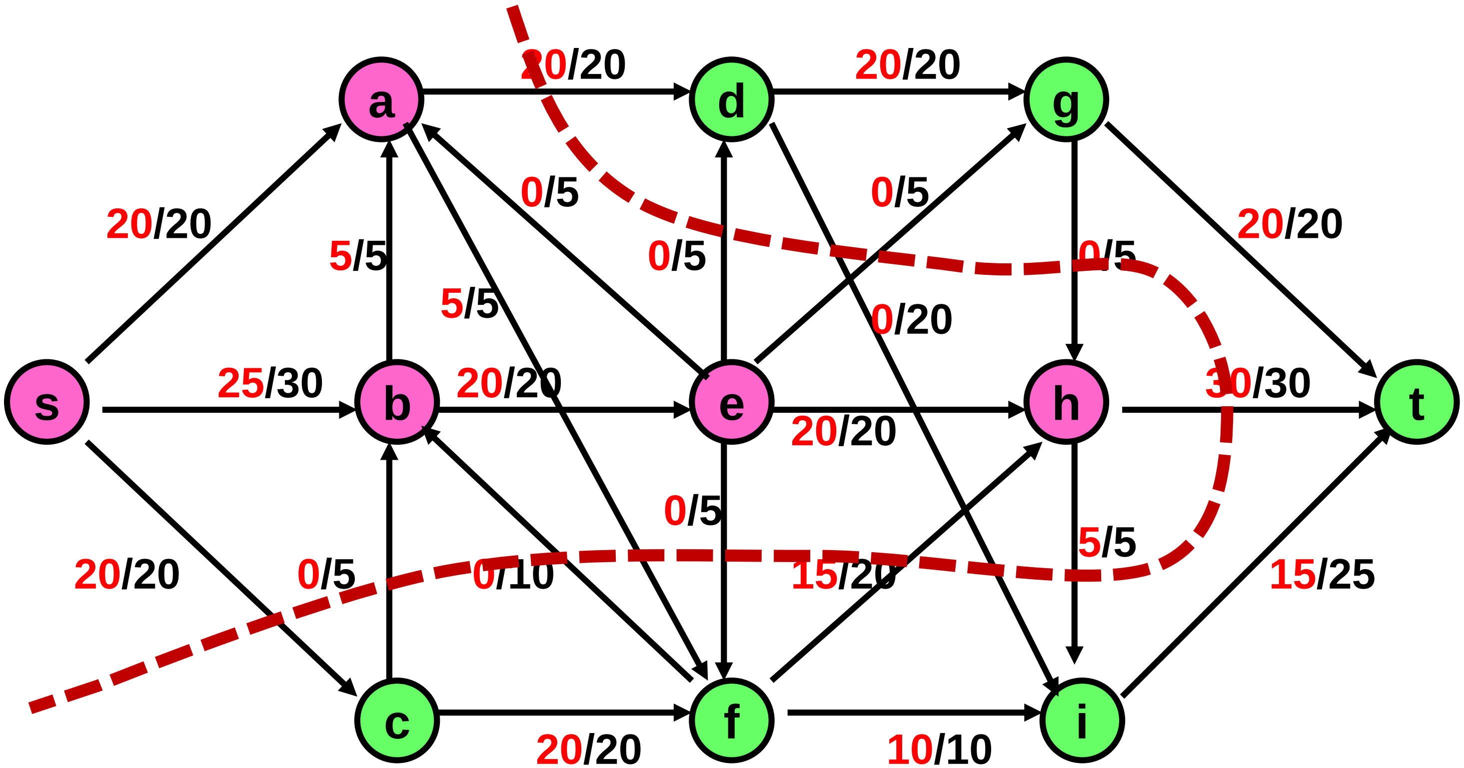
Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Flow(S,T): net flow out of S
  - Sum of flows out of S minus sum of flows into S

- Flow(S,T) <= Cap(S,T)

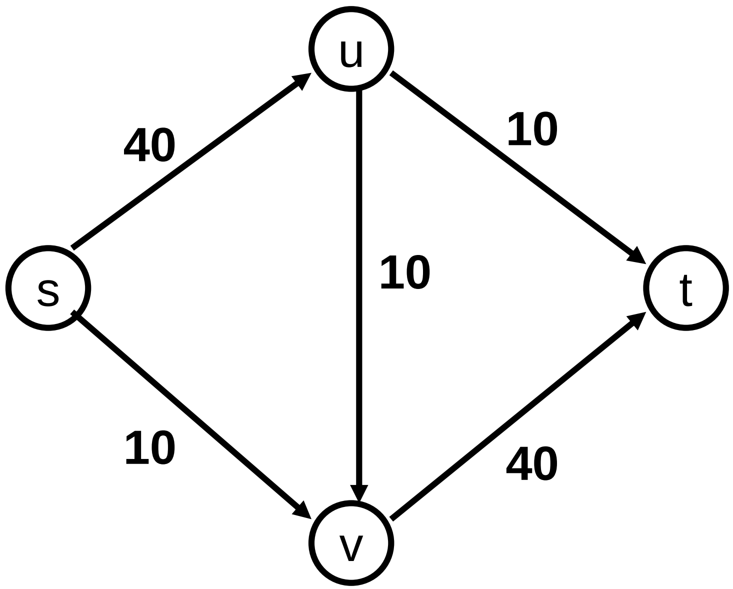
What is Cap(S,T) and Flow(S,T)

S={s, a, b, e, h}, T = {c, f, i, d, g, t}

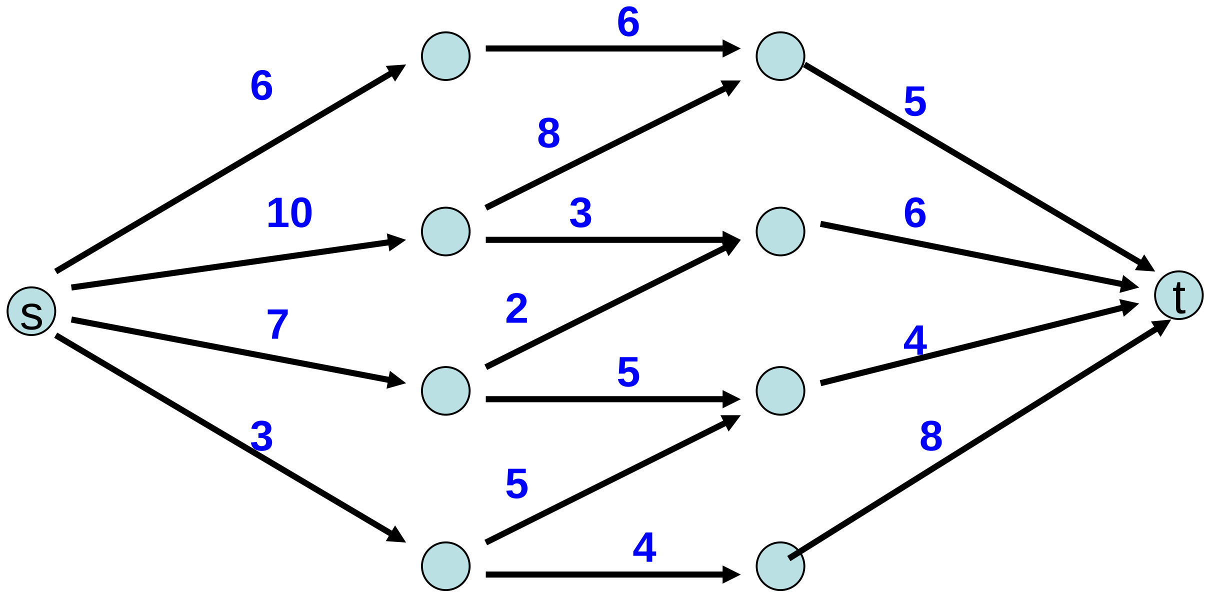


Cap(S,T) = 95, Flow(S,T) = 80 – 15 = 65

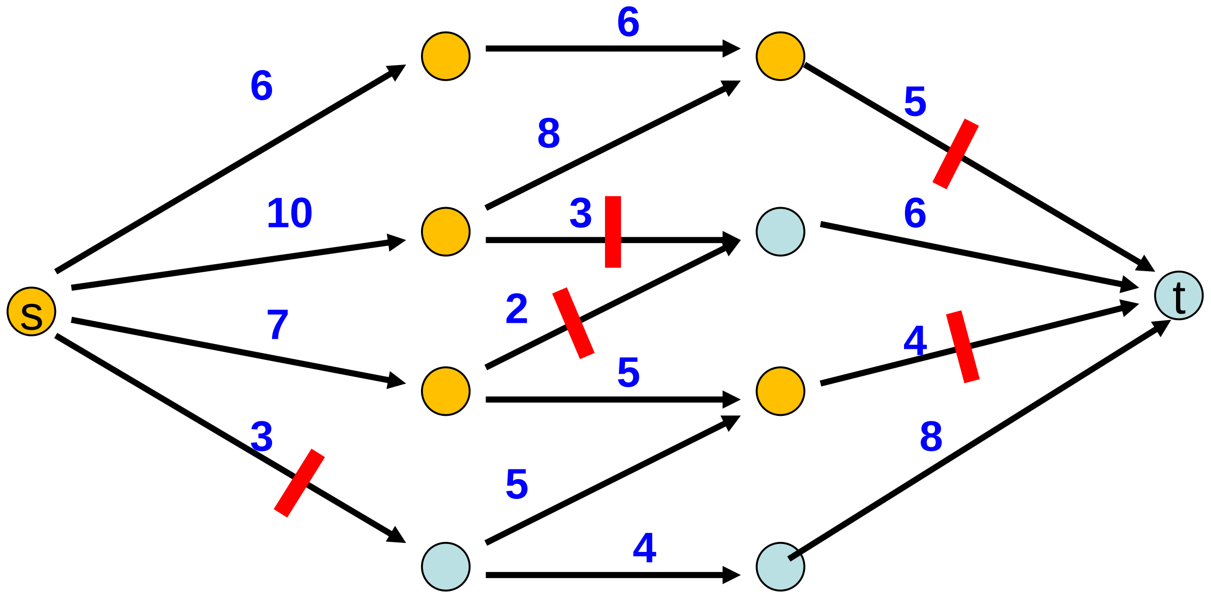
Minimum value cut



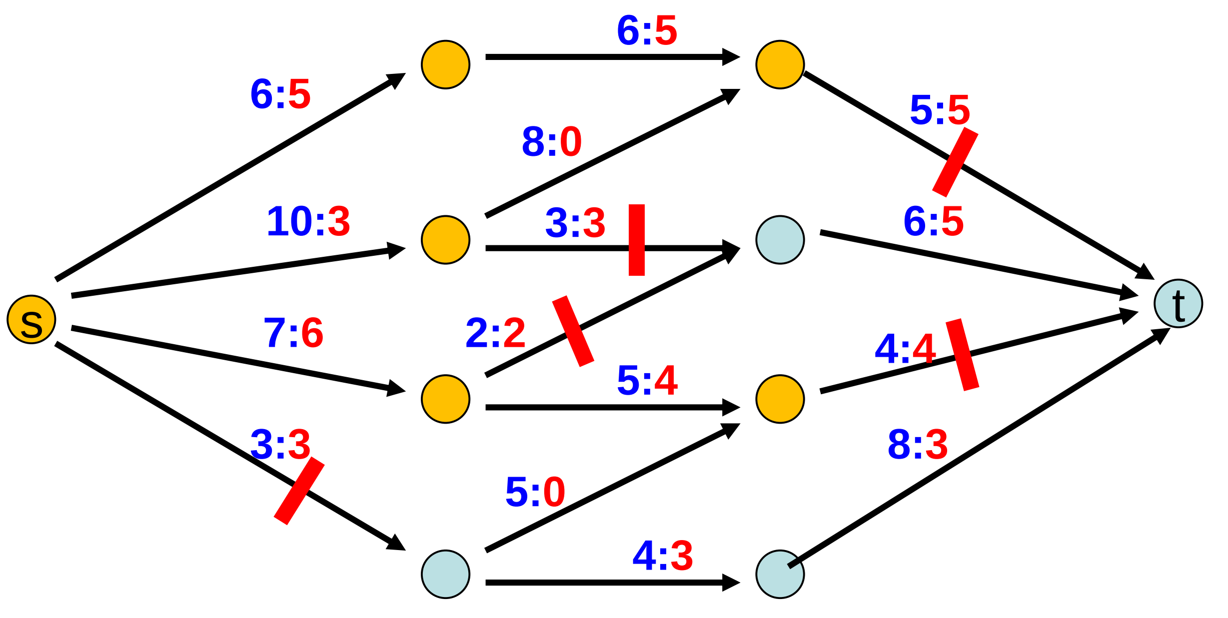
Find a minimum value cut



Find a minimum value cut

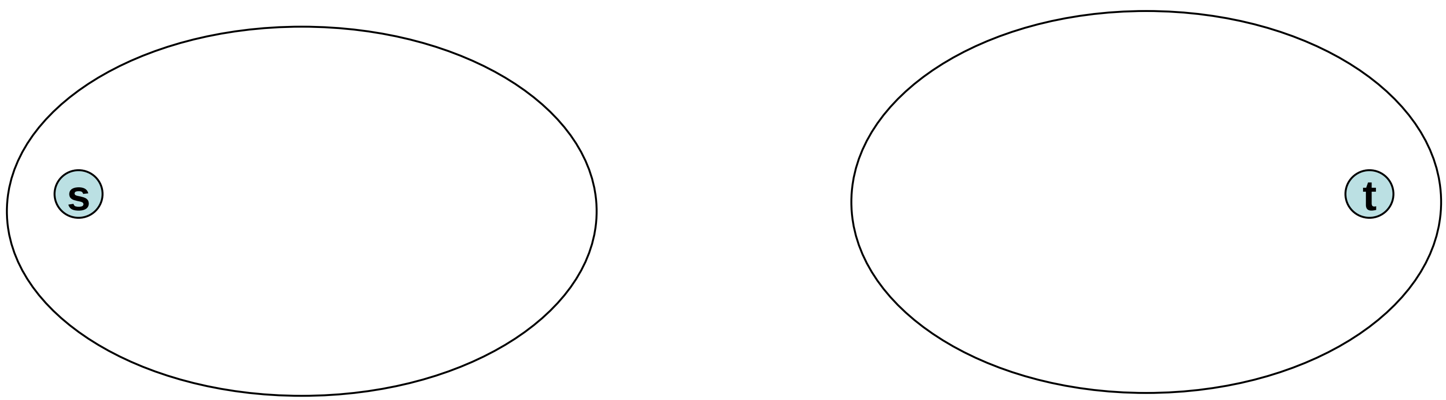


Find a minimum value cut

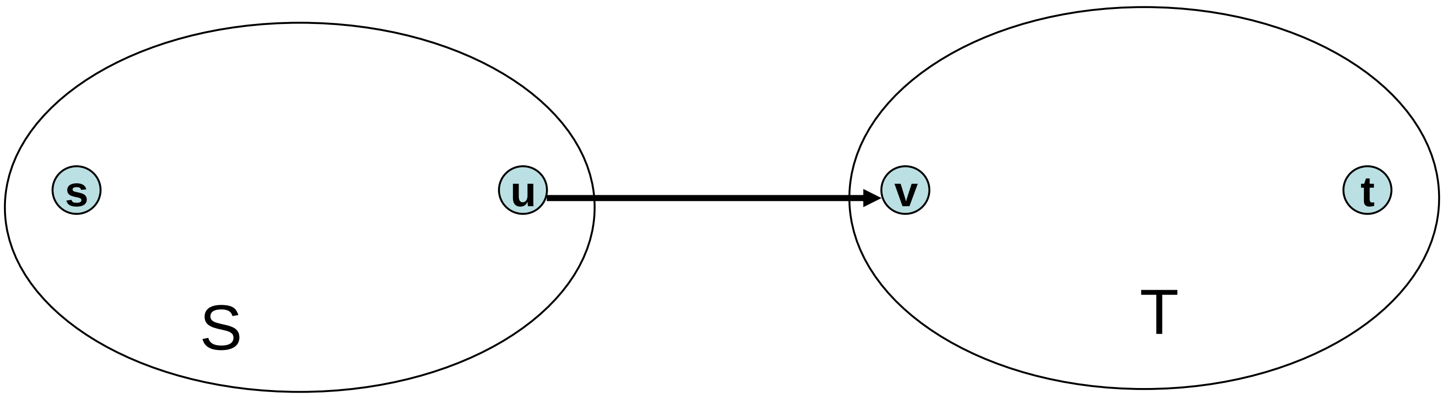


MaxFlow – MinCut Theorem

- There exists a flow which has the same value of the minimum cut
- Proof: Consider a flow where the residual graph has no s-t path with positive capacity
- Let S be the set of vertices in G_R reachable from s with paths of positive capacity



Let S be the set of vertices in G_R reachable from s with paths of positive capacity



What can we say about the flows and capacity between u and v?

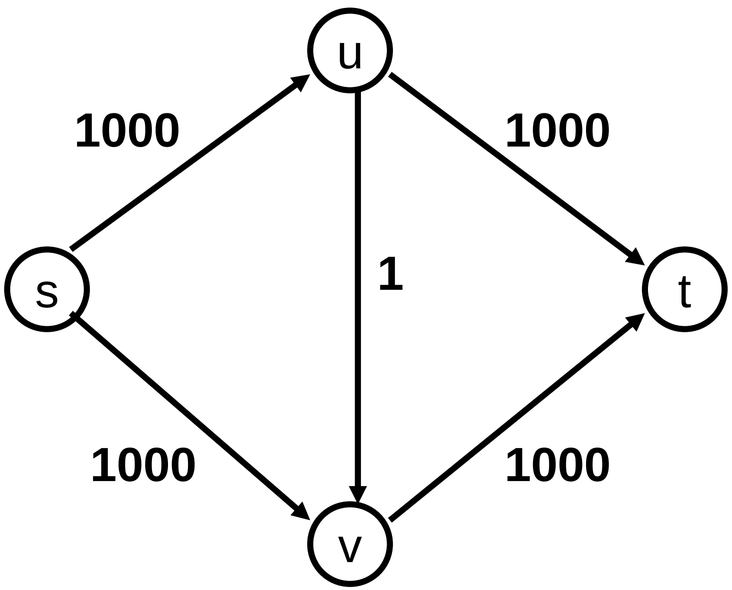
Max Flow - Min Cut Theorem

- Ford-Fulkerson algorithm finds a flow where the residual graph is disconnected, hence FF finds a maximum flow.

- If we want to find a minimum cut, we begin by looking for a maximum flow.

Performance

- The worst case performance of the Ford-Fulkerson algorithm is horrible



Better methods of finding augmenting paths

- Find the maximum capacity augmenting path
  - $O(m^2 \log(C))$ time algorithm for network flow
- Find the shortest augmenting path
  - $O(m^2 n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
  - $O(mn \log n)$ time algorithm for network flow

Problem Reduction

- Reduce Problem A to Problem B
  - Convert an instance of Problem A to an instance of Problem B
  - Use a solution of Problem B to get a solution to Problem A
- Practical
  - Use a program for Problem B to solve Problem A
- Theoretical
  - Show that Problem B is at least as hard as Problem A

Problem Reduction Examples

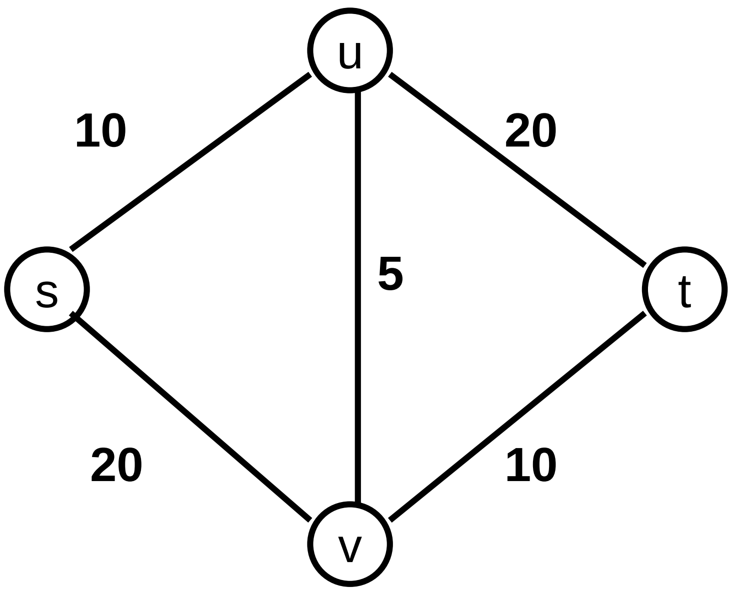
- Reduce the problem of finding the Maximum of a set of integers to finding the Minimum of a set of integers

Find the maximum of: 8, -3, 2, 12, 1, -6

Construct an equivalent minimization problem

Undirected Network Flow

- Undirected graph with edge capacities
- Flow may go either direction along the edges (subject to the capacity constraints)



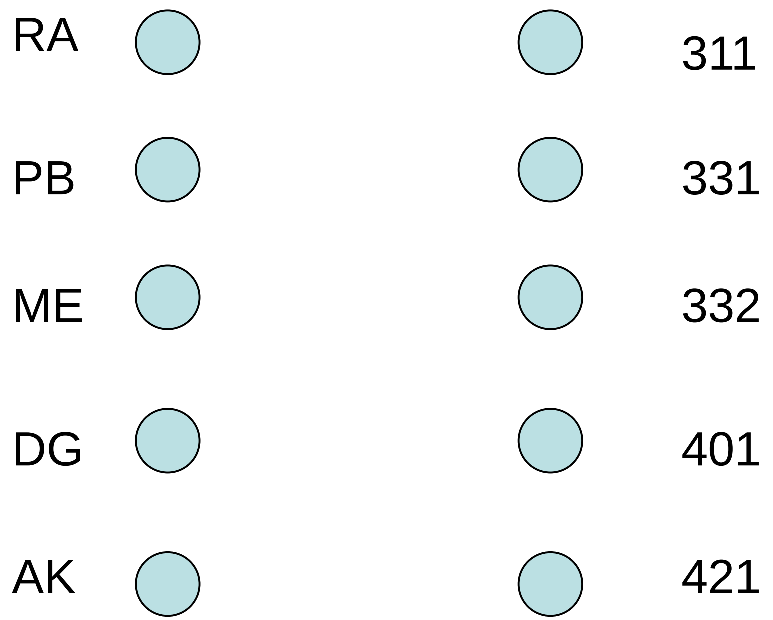
Construct an equivalent flow problem

Bipartite Matching

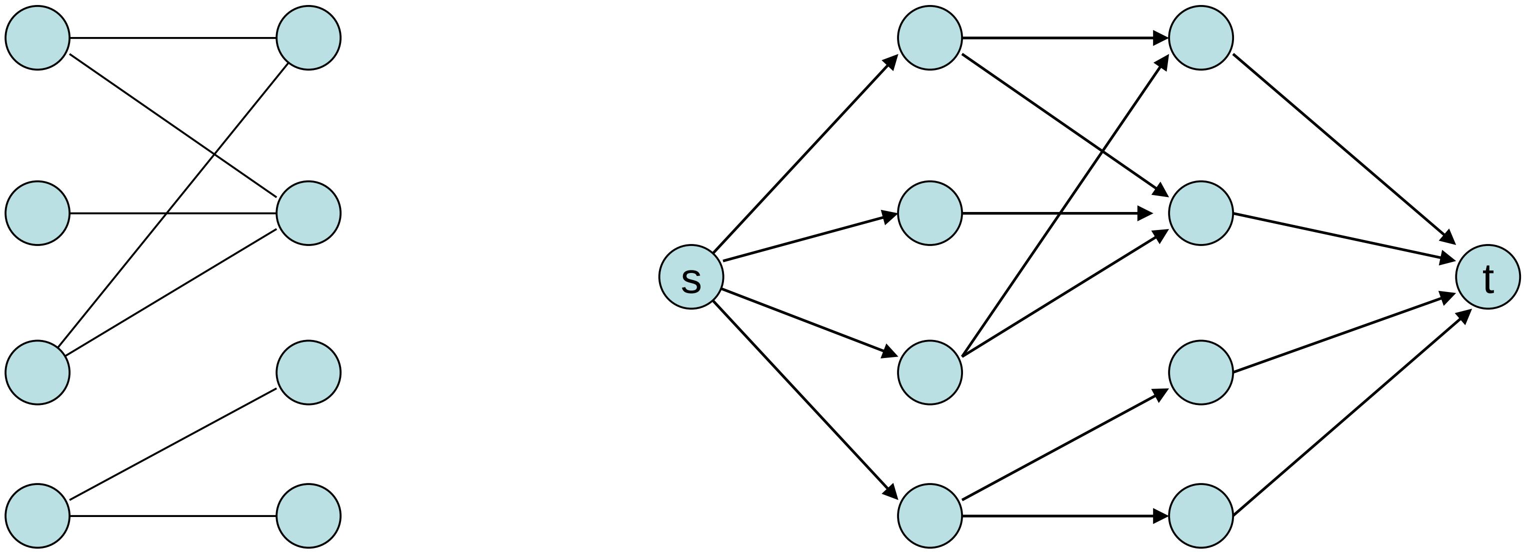
- A graph $G=(V,E)$ is bipartite if the vertices can be partitioned into disjoints sets X,Y
- A matching M is a subset of the edges that does not share any vertices
- Find a matching as large as possible

Application

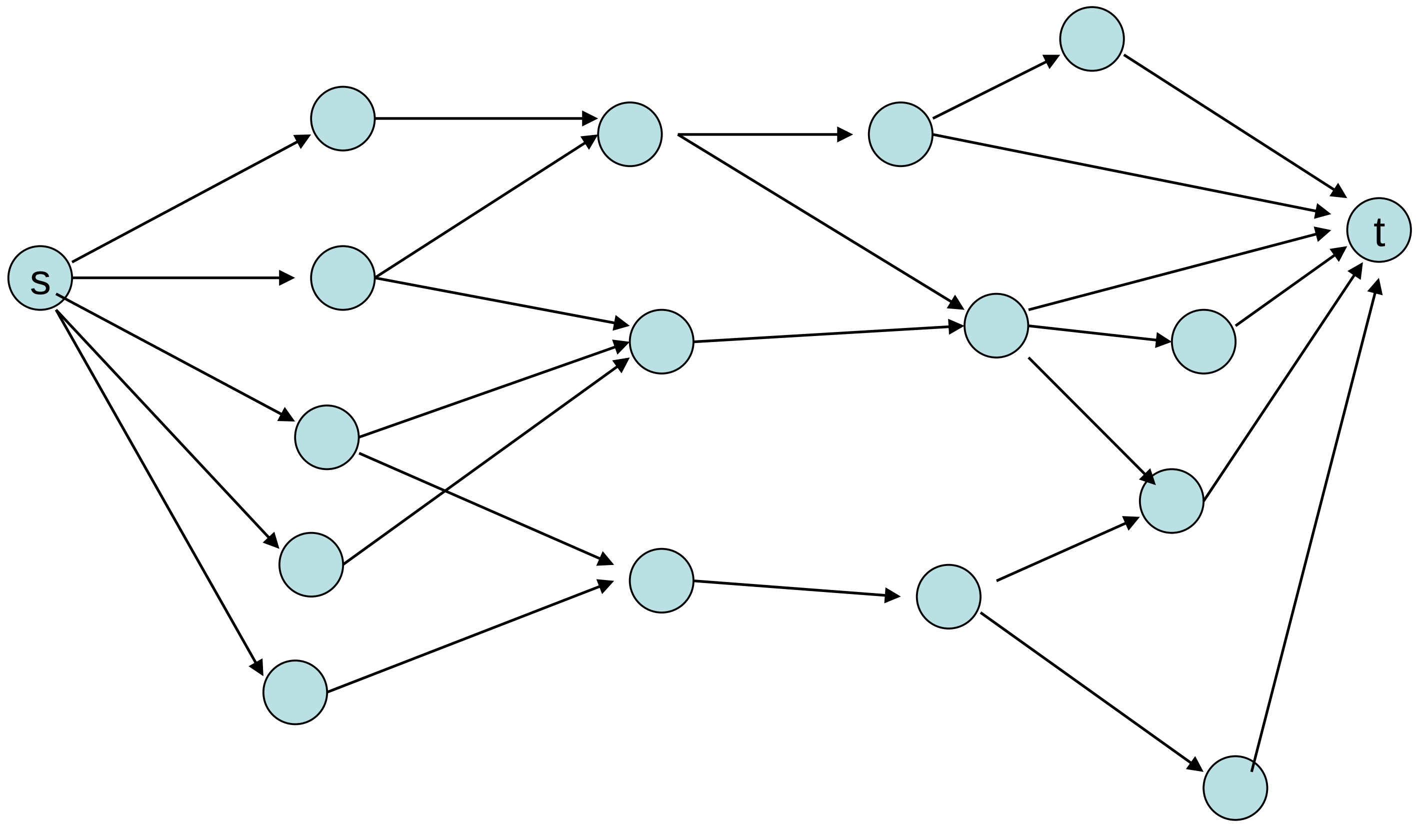
- A collection of teachers
- A collection of courses
- And a graph showing which teachers can teach which courses



Converting Matching to Network Flow



Finding edge disjoint paths



Construct a maximum cardinality set of edge disjoint paths