CSE 421
Algorithms
Lecture 21
Network Flow, Part 1

Announcements
• Sample finals posted
• No questions planned on NP-completeness

Network Flow
Outline
• Network flow definitions
• Flow examples
• Augmenting Paths
• Residual Graph
• Ford Fulkerson Algorithm
• Cuts
• Maxflow-MinCut Theorem

Network Flow Definitions
• Capacity
• Source, Sink
• Capacity Condition
• Conservation Condition
• Value of a flow

Flow Example
Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, c(e) >= 0
- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e)
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
    - The flow leaving the source is as large as possible

Flow Example

Find a maximum flow

Residual Graph

- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph G_R
  - G: edge e from u to v with capacity c and flow f
  - G_R: edge e' from u to v with capacity c - f
  - G_R: edge e'' from v to u with capacity f

Flow assignment and the residual graph
Augmenting Path Algorithm

- Augmenting path
  - Vertices $v_1, v_2, \ldots, v_k$
  - $v_1 = s$, $v_k = t$
  - Possible to add $b$ units of flow between $v_j$ and $v_{j+1}$ for $j = 1 \ldots k-1$

Build the residual graph

Find two augmenting paths

Augmenting Path Lemma

- Let $P = v_1, v_2, \ldots, v_k$ be a path from $s$ to $t$ with minimum capacity $b$ in the residual graph.
- $b$ units of flow can be added along the path $P$ in the flow graph.

Proof

- Add $b$ units of flow along the path $P$
- What do we need to verify to show we have a valid flow after we do this?

Ford-Fulkerson Algorithm (1956)

while not done
  Construct residual graph $G_R$
  Find an $s$-$t$ path $P$ in $G_R$ with capacity $b > 0$
  Add $b$ units along in $G$

If the sum of the capacities of edges leaving $S$ is at most $C$, then the algorithm takes at most $C$ iterations
Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- \( \text{Cap}(S,T) \): sum of the capacities of edges from S to T
- \( \text{Flow}(S,T) \): net flow out of S
  - Sum of flows out of S minus sum of flows into S
- \( \text{Flow}(S,T) \leq \text{Cap}(S,T) \)

What is Cap(S,T) and Flow(S,T)

\[ S = \{s, a, b, e, h\}, \quad T = \{c, f, i, d, g, t\} \]

\( \text{Cap}(S,T) = 95, \quad \text{Flow}(S,T) = 80 - 15 = 65 \)

Minimum value cut

Find a minimum value cut
Find a minimum value cut

MaxFlow – MinCut Theorem

- There exists a flow which has the same value of the minimum cut
- Proof: Consider a flow where the residual graph has no s-t path with positive capacity
- Let S be the set of vertices in $G_R$ reachable from s with paths of positive capacity

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Max Flow - Min Cut Theorem

- Ford-Fulkerson algorithm finds a flow where the residual graph is disconnected, hence FF finds a maximum flow.
- If we want to find a minimum cut, we begin by looking for a maximum flow.