CSE 421 Algorithms

Richard Anderson

Lecture 20

Shortest Paths and Network Flow

1

Announcements

- HW 8 Available
  - Due Friday, March 15
- Final Exam
  - 2:30-4:20 pm, Monday, March 18, THO 101

2

Shortest Paths with Dynamic Programming

3

Shortest Path Problem

- Dijkstra's Single Source Shortest Paths Algorithm
  - O(mlog n) time, positive cost edges
- Bellman-Ford Algorithm
  - O(mn) time for graphs which can have negative cost edges

4

Lemma

- If a graph has no negative cost cycles, then the shortest paths are simple paths
- Shortest paths have at most n-1 edges

5

Shortest paths with a fixed number of edges

- Find the shortest path from s to w with exactly k edges

6

Express as a recurrence

- Compute distance from starting vertex s
- $Opt_k(w) = \min_x [Opt_{k-1}(x) + c_{xw}]$
- $Opt_0(w) = 0$ if w = s and infinity otherwise

7

Algorithm, Version 1

```
for each w
   M[0, w] = infinity;
M[0, s] = 0;
for i = 1 to n-1
   for each w
      M[i, w] = min_x(M[i-1,x] + cost[x,w]);
```

8

Algorithm, Version 2

```
for each w
   M[0, w] = infinity;
M[0, s] = 0;
for i = 1 to n-1
   for each w
      M[i, w] = min(M[i-1, w], min_x(M[i-1,x] + cost[x,w]))
```

9

Algorithm, Version 3

```
for each w
   M[w] = infinity;
M[s] = 0;
for i = 1 to n-1
   for each w
      M[w] = min(M[w], min_x(M[x] + cost[x,w]))
```

10

Correctness Proof for Algorithm 3

- Key lemma – at the end of iteration i, for all w, M[w] <= M[i, w];

11

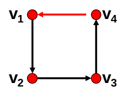
Algorithm, Version 4

```
for each w
   M[w] = infinity;
M[s] = 0;
for i = 1 to n-1
   for each w
      for each x
         if (M[w] > M[x] + cost[x,w])
            P[w] = x
            M[w] = M[x] + cost[x,w]
```

12

If the pointer graph has a cycle, then the graph has a negative cost cycle

- If P[w] = x then M[w] >= M[x] + cost(x,w)
  - Equal when w is updated
  - M[x] could be reduced after update
- Let $v_1, v_2, \ldots v_k$ be a cycle in the pointer graph with (v_k,v_1) the last edge added
  - Just before the update
    - $M[v_j] >= M[v_{j+1}] + cost(v_{j+1}, v_j)$ for $j < k$
    - $M[v_k] > M[v_1] + cost(v_1, v_k)$
  - Adding everything up
    - $0 > cost(v_1,v_2) + cost(v_2,v_3) + \ldots + cost(v_k, v_1)$



13

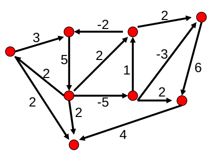
Negative Cycles

- If the pointer graph has a cycle, then the graph has a negative cycle
- Therefore: if the graph has no negative cycles, then the pointer graph has no negative cycles

14

Finding negative cost cycles

- What if you want to find negative cost cycles?



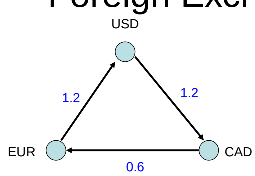
15

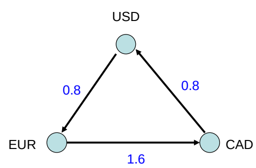
What about finding Longest Paths

- Can we just change Min to Max?

16

Foreign Exchange Arbitrage

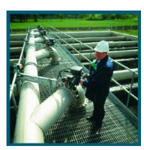




	USD	EUR	CAD
USD	------	0.8	1.2
EUR	1.2	------	1.6
CAD	0.8	0.6	-----

17

Network Flow



18

Outline

- Network flow definitions
- Flow examples
- Augmenting Paths
- Residual Graph
- Ford Fulkerson Algorithm
- Cuts
- Maxflow-MinCut Theorem

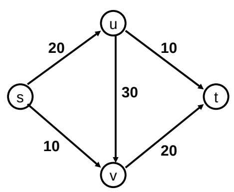
19

Network Flow Definitions

- Capacity
- Source, Sink
- Capacity Condition
- Conservation Condition
- Value of a flow

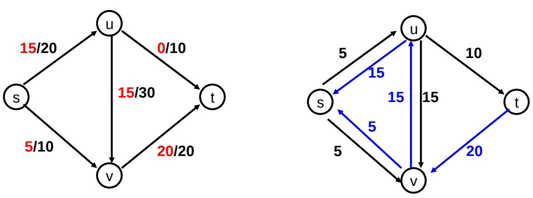
20

Flow Example



21

Flow assignment and the residual graph



22

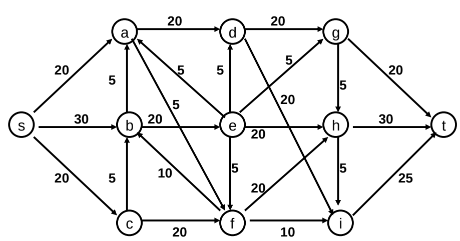
Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, c(e) >= 0
- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e)
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible

23

Find a maximum flow

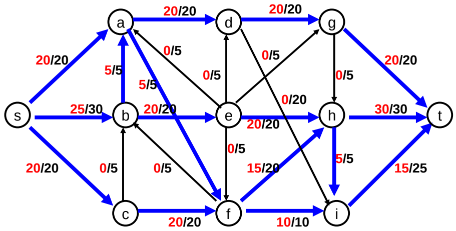
Value of flow:



Construct a maximum flow and indicate the flow value

24

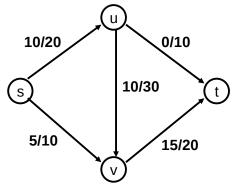
Find a maximum flow



25

Augmenting Path Algorithm

- Augmenting path
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
    - Possible to add b units of flow between v_j and v_{j+1} for j = 1 … k-1



26