CSE 421
Algorithms

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Divide and Conquer

Announcements

What you really need to know about recurrences

- Work per level changes geometrically with the level
- Geometrically increasing (x > 1)
  - The bottom level wins
- Geometrically decreasing (x < 1)
  - The top level wins
- Balanced (x = 1)
  - Equal contribution

$T(n) = aT(n/b) + n^c$

- Balanced: $a = b^c$
  - $T(n) = 4T(n/2) + n^2$
- Increasing: $a > b^c$
  - $T(n) = 9T(n/8) + n$
  - $T(n) = 3T(n/4) + n^{1/2}$
- Decreasing: $a < b^c$
  - $T(n) = 5T(n/8) + n$
  - $T(n) = 7T(n/2) + n^3$

Divide and Conquer Algorithms

- Split into sub problems
- Recursively solve the problem
- Combine solutions

- Make progress in the split and combine stages
  - Quicksort – progress made at the split step
  - Mergesort – progress made at the combine step
- D&C Algorithms
  - Strassen's Algorithm – Matrix Multiplication
  - Inversions
  - Median
  - Closest Pair
  - Integer Multiplication
  - FFT

How to multiply 2 x 2 matrices with 7 multiplications

Multiply 2 x 2 Matrices:

| r s | | a b| |e g|
| t u| = | c d| | f h|

$r = p_1 + p_2 – p_4 + p_6$

$s = p_4 + p_5$

$t = p_6 + p_7$

$u = p_2 - p_3 + p_5 - p_7$

Where:

$p_1 = (b – d)(f + h)$

$p_2 = (a + d)(e + h)$

$p_3 = (a – c)(e + g)$

$p_4 = (a + b)h$

$p_5 = a(g – h)$

$p_6 = d(f – e)$

$p_7 = (c + d)e$

Corrected version from AHU 1974

Strassen's Algorithms

- Treat n x n matrices as 2 x 2 matrices of n/2 x n/2 submatrices
- Use Strassen's trick to multiply 2 x 2 matrices with 7 multiplies
- Base case standard multiplication for single entries
- Recurrence: $T(n) = 7\,T(n/2) + cn^2$
- Solution is $O(7^{\log n}) = O(n^{\log 7})$ which is about $O(n^{2.807})$

Inversion Problem

- Let $a_1, \ldots, a_n$ be a permutation of 1 . . n
- (a_i, a_j) is an inversion if $i < j$ and $a_i > a_j$

4, 6, 1, 7, 3, 2, 5

- Problem: given a permutation, count the number of inversions
- This can be done easily in $O(n^2)$ time
  - Can we do better?

Application

- Counting inversions can be use to measure how close ranked preferences are
  - People rank 20 movies, based on their rankings you cluster people who like that same type of movie

Counting Inversions

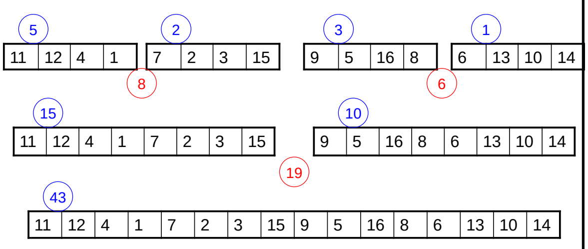
11	12	4	1	7	2	3	15	9	5	16	8	6	13	10	14

Count inversions on lower half

Count inversions on upper half

Count the inversions between the halves

Count the Inversions



Problem – how do we count inversions between sub problems in O(n) time?

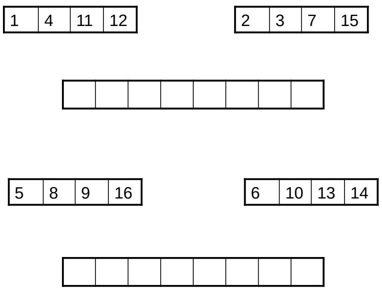
- Solution – Count inversions while merging

1	2	3	4	7	11	12	15

5	6	8	9	10	13	14	16

Standard merge algorithm – add to inversion count when an element is moved from the upper array to the solution

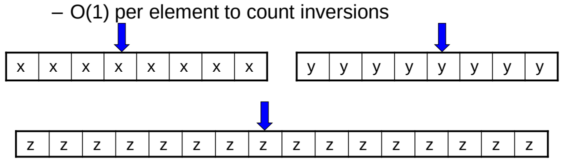
Use the merge algorithm to count inversions



Indicate the number of inversions for each element detected when merging

Inversions

- Counting inversions between two sorted lists
  - O(1) per element to count inversions



- Algorithm summary
  - Satisfies the "Standard recurrence"
  - $T(n) = 2\,T(n/2) + cn$

Computing the Median

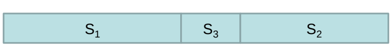
- Given n numbers, find the number of rank n/2
- One approach is sorting
  - Sort the elements, and choose the middle one
  - Can you do better?

Problem generalization

- *Selection*, given n numbers and an integer k, find the k-th largest

Select(A, k)

```
Select(A, k){
        Choose element x from A
        S1 = {y in A | y < x}
        S2 = {y in A | y > x}
        S3 = {y in A | y = x}
        if (|S2| >= k)
                return Select(S2, k)
        else if (|S2| + |S3| >= k)
                return x
        else
                return Select(S1, k - |S2| - |S3|)
}
```



Randomized Selection

- Choose the element at random
- Analysis can show that the algorithm has expected run time O(n)

Deterministic Selection

- What is the run time of select if we can guarantee that choose finds an x such that $|S_1| < 3n/4$ and $|S_2| < 3n/4$ in O(n) time

BFPRT Algorithm





- A very clever choose algorithm . . .

Split into n/5 sets of size 5
M be the set of medians of these sets
Let x be the median of M

BFPRT runtime

$|S_1| < 3n/4$, $|S_2| < 3n/4$

Split into n/5 sets of size 5
M be the set of medians of these sets
x be the median of M
Construct S_1 and S_2
Recursive call in S_1 or S_2

BFPRT Recurrence

- $T(n) <= T(3n/4) + T(n/5) + c\ n$

Prove that $T(n) <= 20\ c\ n$