CSE 421
Algorithms

Winter 2019

Lecture 11

Minimum Spanning Trees (Part II)

Interval Scheduling

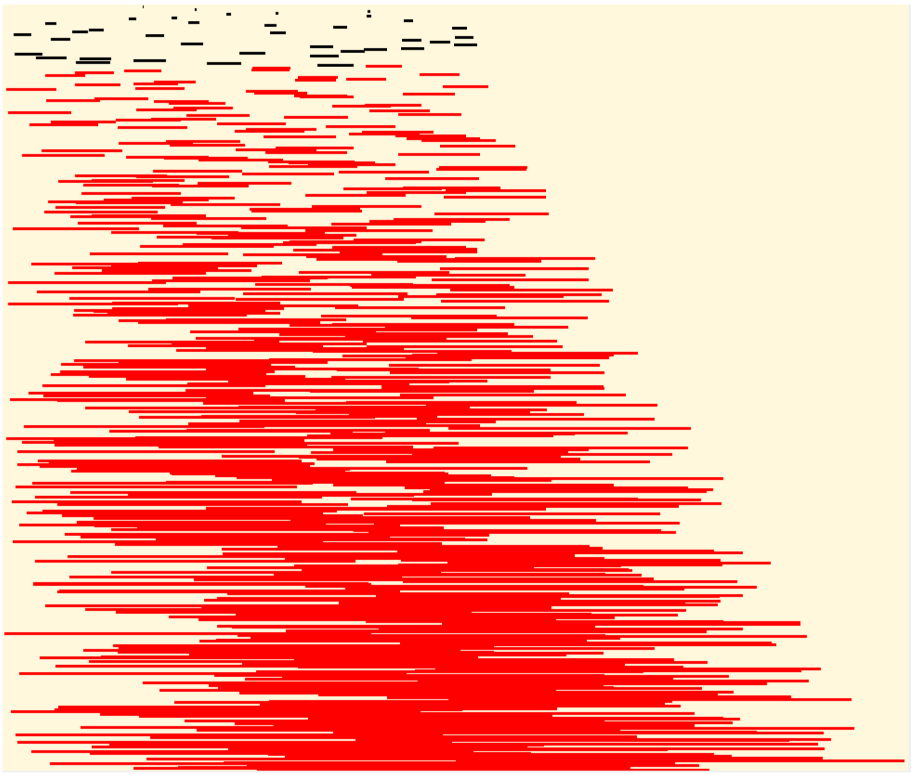
- What is the expected size of the maximum independent set for random intervals
- What is the expected size of the maximum intersection for random intervals

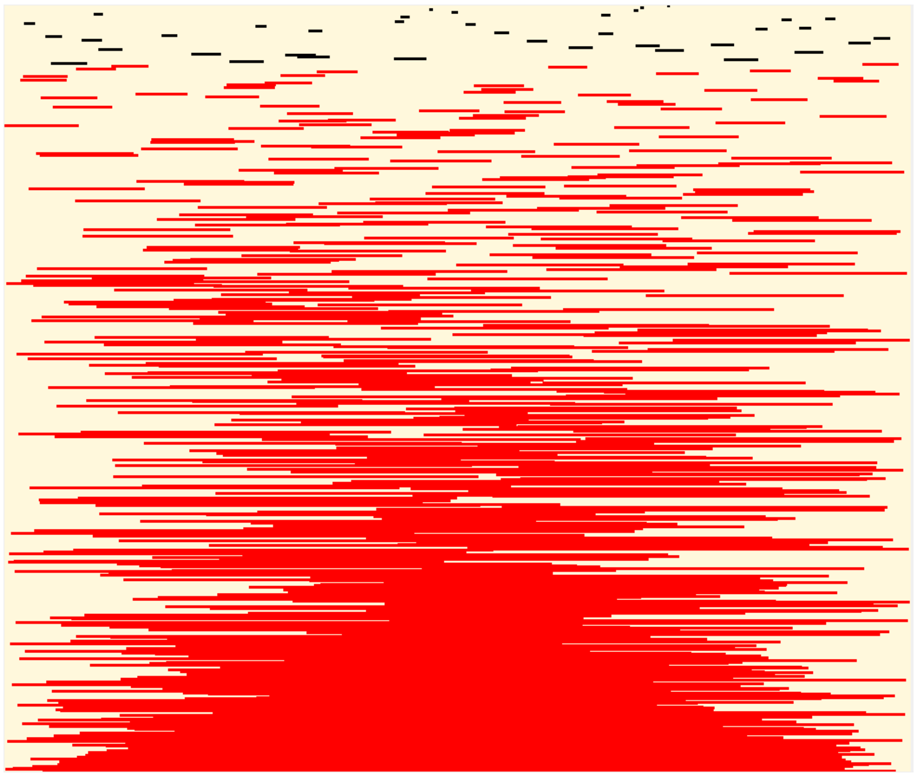
Method 1: Each interval assigned a random start position and random length from [0,1]

Method 2: Random permutation of interval endpoints

Independent Set

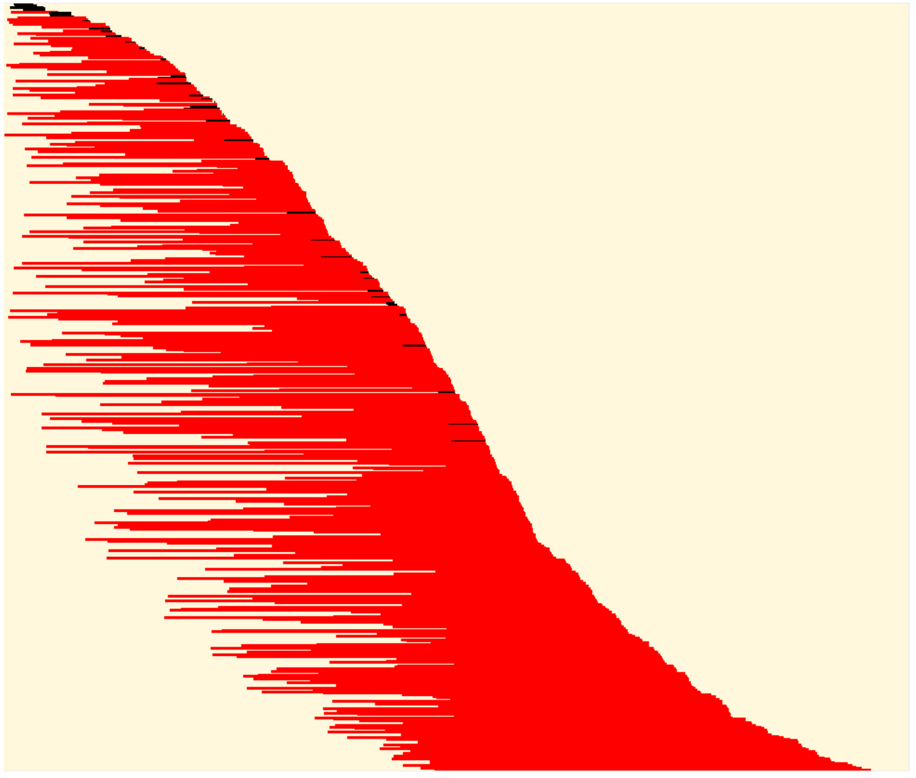
Maximum independent set only contains small intervals

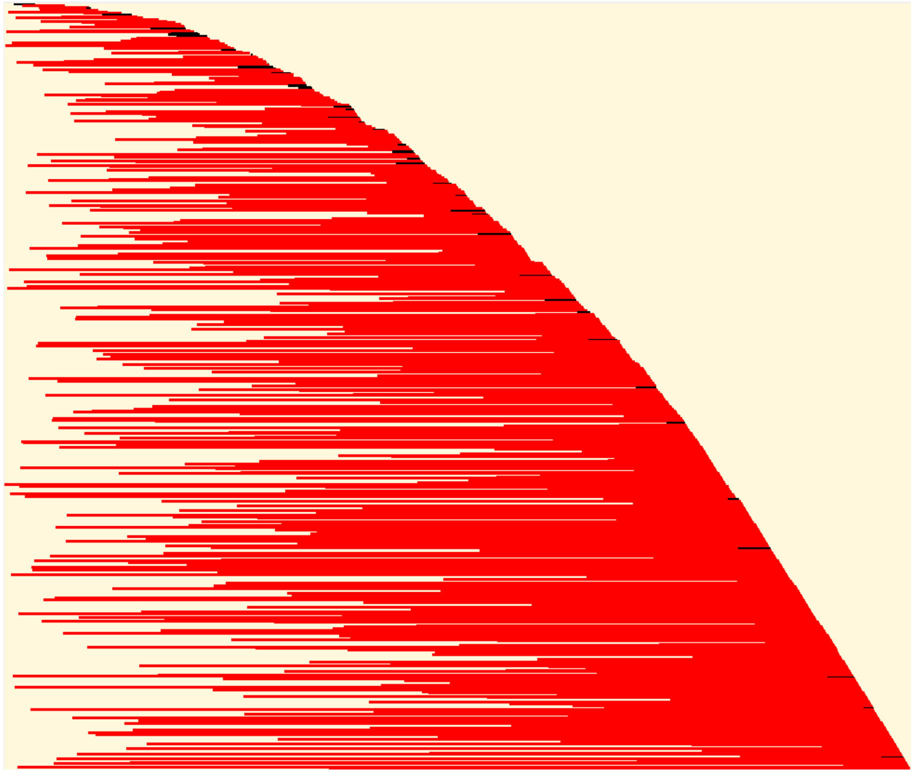




Maximum Intersection

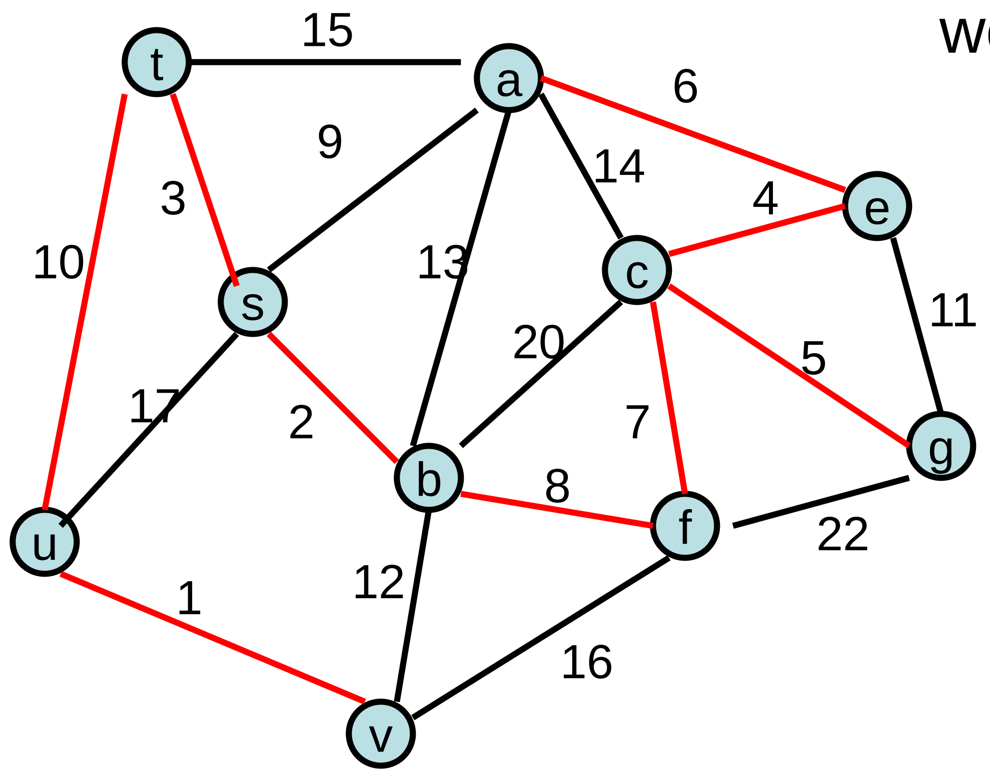
- Maximum intersection is at the middle





Minimum Spanning Tree

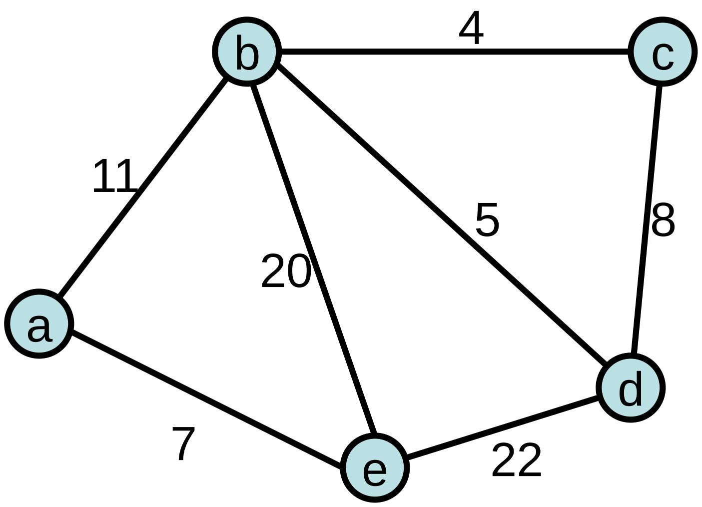
Undirected Graph G=(V,E) with edge weights



For simplicity, assume all edge costs are distinct

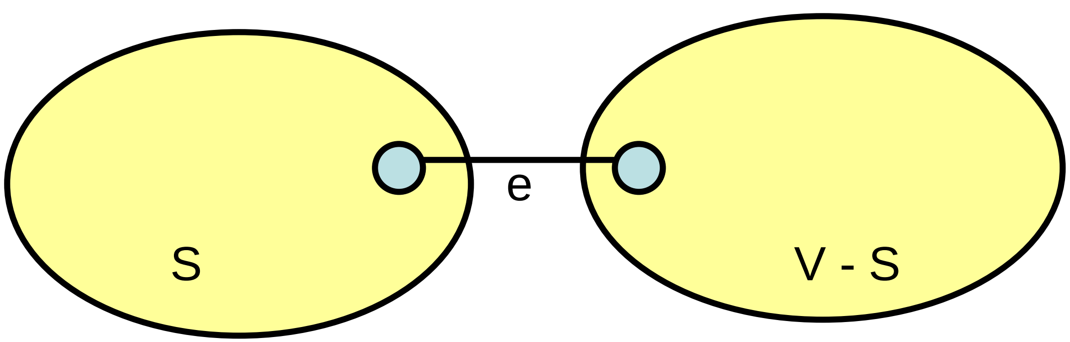
Greedy Algorithms for Minimum Spanning Tree

- [Prim] Extend a tree by including the cheapest out going edge
- [Kruskal] Add the cheapest edge that joins disjoint components
- [ReverseDelete] Delete the most expensive edge that does not disconnect the graph



Edge inclusion lemma

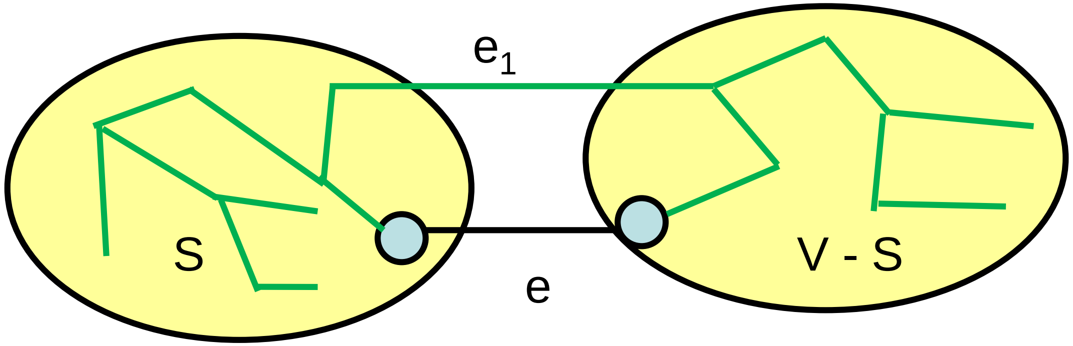
- Let S be a subset of V, and suppose e = (u, v) is the minimum cost edge of E, with u in S and v in V-S
- e is in every minimum spanning tree of G
  - Or equivalently, if e is not in T, then T is not a minimum spanning tree



e is the minimum cost edge between S and V-S

Proof

- Suppose T is a spanning tree that does not contain e
- Add e to T, this creates a cycle
- The cycle must have some edge $e_1 = (u_1, v_1)$ with u_1 in S and v_1 in V-S



- $T_1 = T - \{e_1\} + \{e\}$ is a spanning tree with lower cost
- Hence, T is not a minimum spanning tree

Optimality Proofs

- Prim's Algorithm computes a MST
- Kruskal's Algorithm computes a MST

- Show that when an edge is added to the MST by Prim or Kruskal, the edge is the minimum cost edge between S and V-S for some set S.

Prim's Algorithm

S = { }; T = { };

while S != V

choose the minimum cost edge
e = (u,v), with u in S, and v in V-S

add e to T

add v to S

Prove Prim's algorithm computes an MST

- Show an edge e is in the MST when it is added to T

Kruskal's Algorithm

Let $C = \{\{v_1\}, \{v_2\}, . . ., \{v_n\}\}$; $T = \{ \}$

while $|C| > 1$

Let $e = (u, v)$ with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C

Replace C_i and C_j by $C_i \cup C_j$

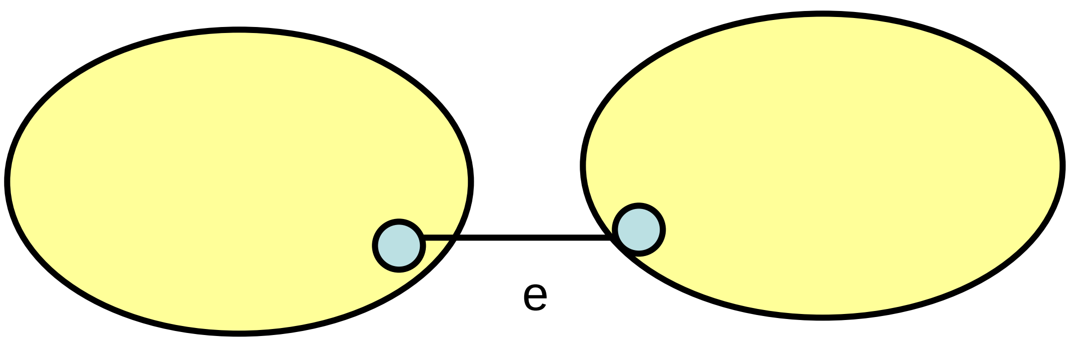
Add e to T

Prove Kruskal's algorithm computes an MST

- Show an edge e is in the MST when it is added to T

Reverse-Delete Algorithm

- Lemma: The most expensive edge on a cycle is never in a minimum spanning tree



Reverse-Delete Algorithm

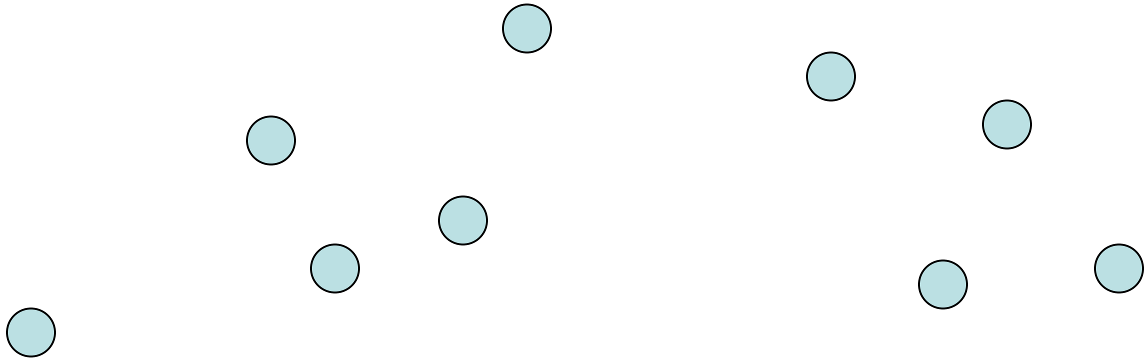
- Let e be the max cost edge whose removal does not disconnect the graph
- Let T be a spanning tree of G=(V, E – {e})

Dealing with the assumption of no equal weight edges

- Force the edge weights to be distinct
  - Add small quantities to the weights
  - Give a tie breaking rule for equal weight edges

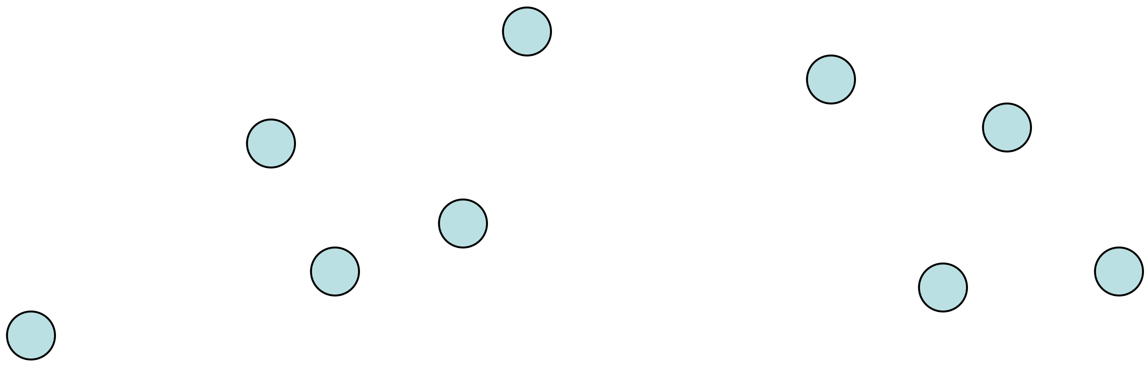
Application: Clustering

- Given a collection of points in an r-dimensional space, and an integer K, divide the points into K sets that are closest together

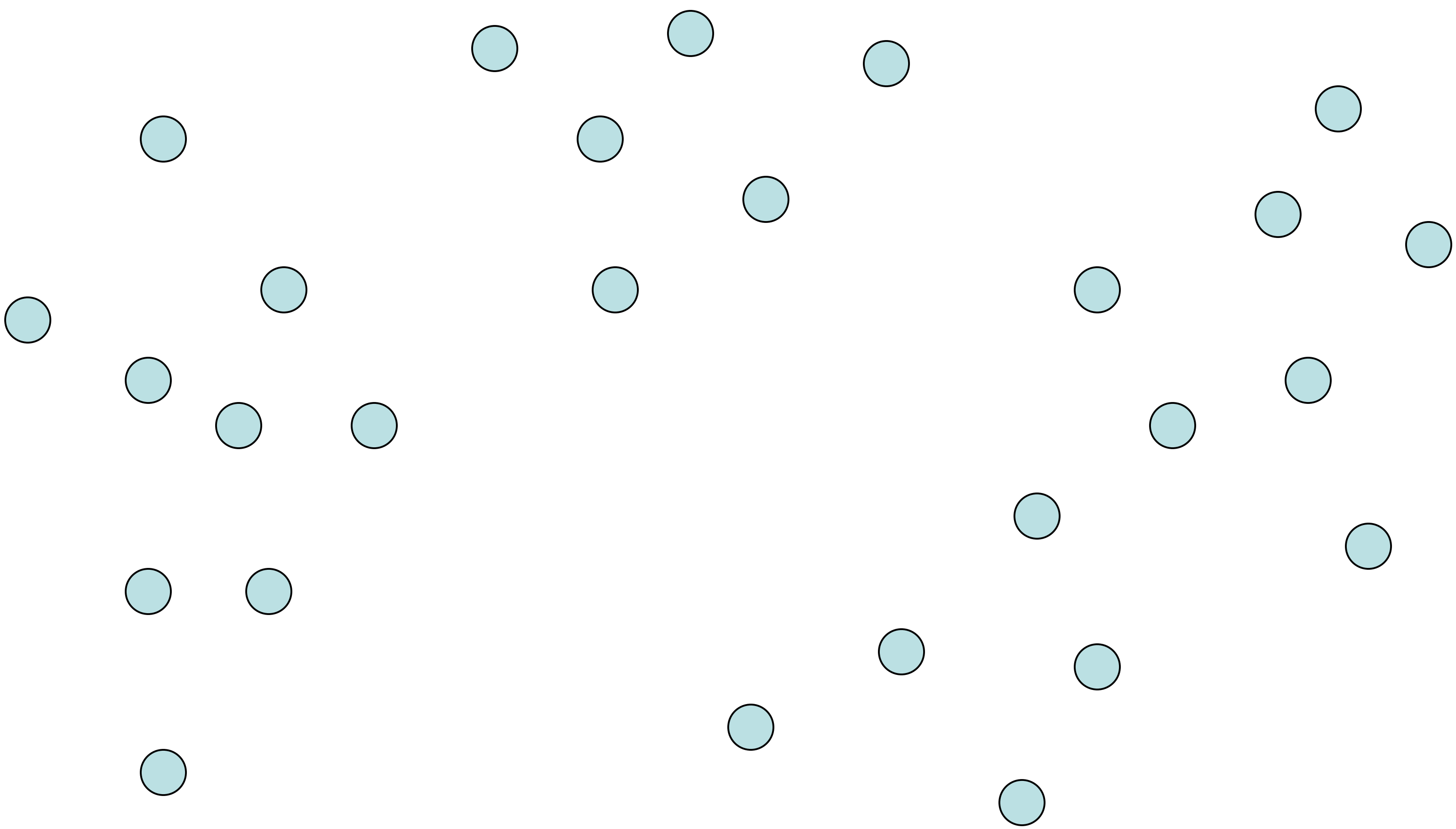


Distance clustering

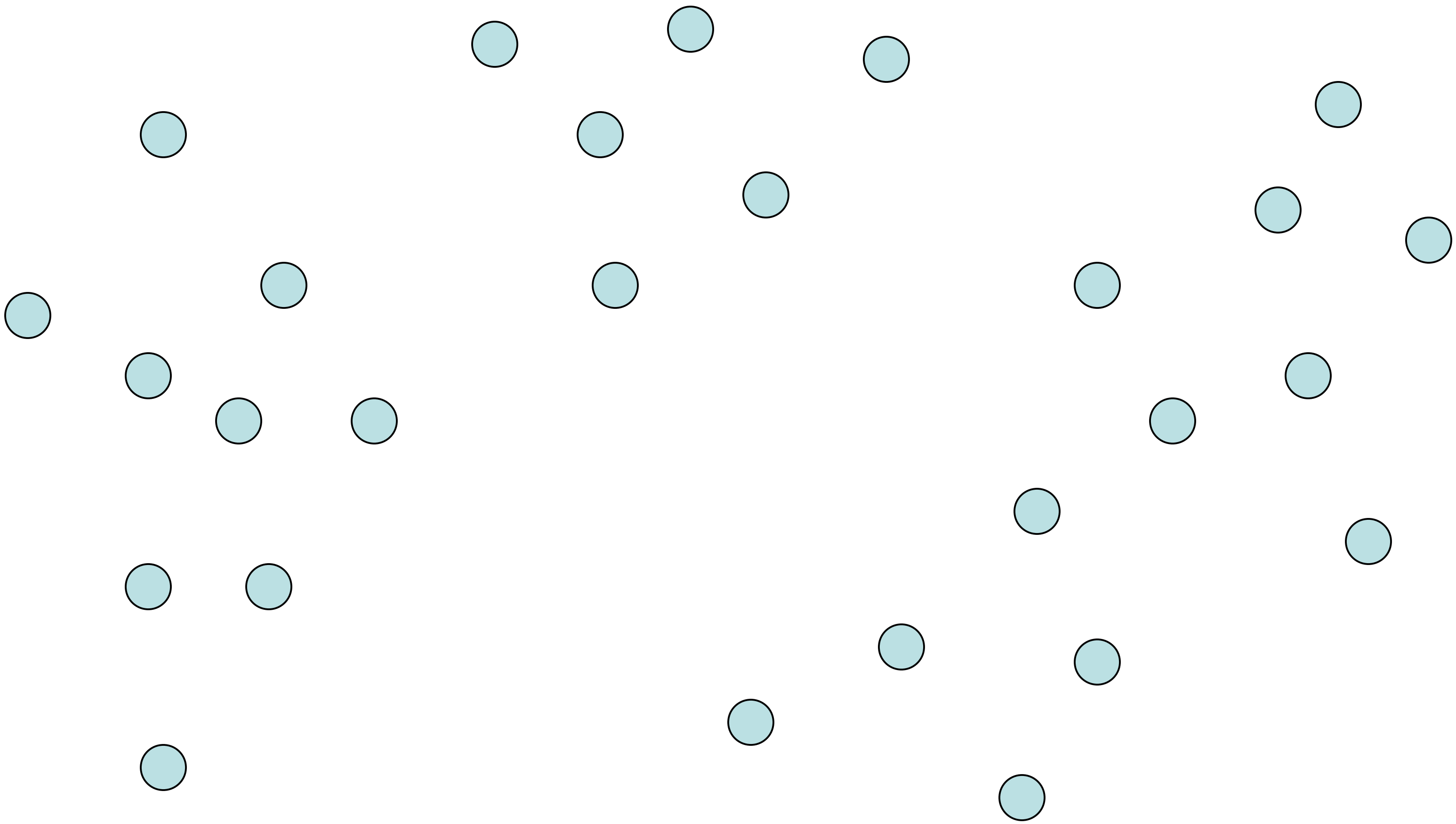
- Divide the data set into K subsets to maximize the distance between any pair of sets
  - dist $(S_1, S_2) = \min \{\mathrm{dist}(x, y) \mid x \text{ in } S_1, y \text{ in } S_2\}$



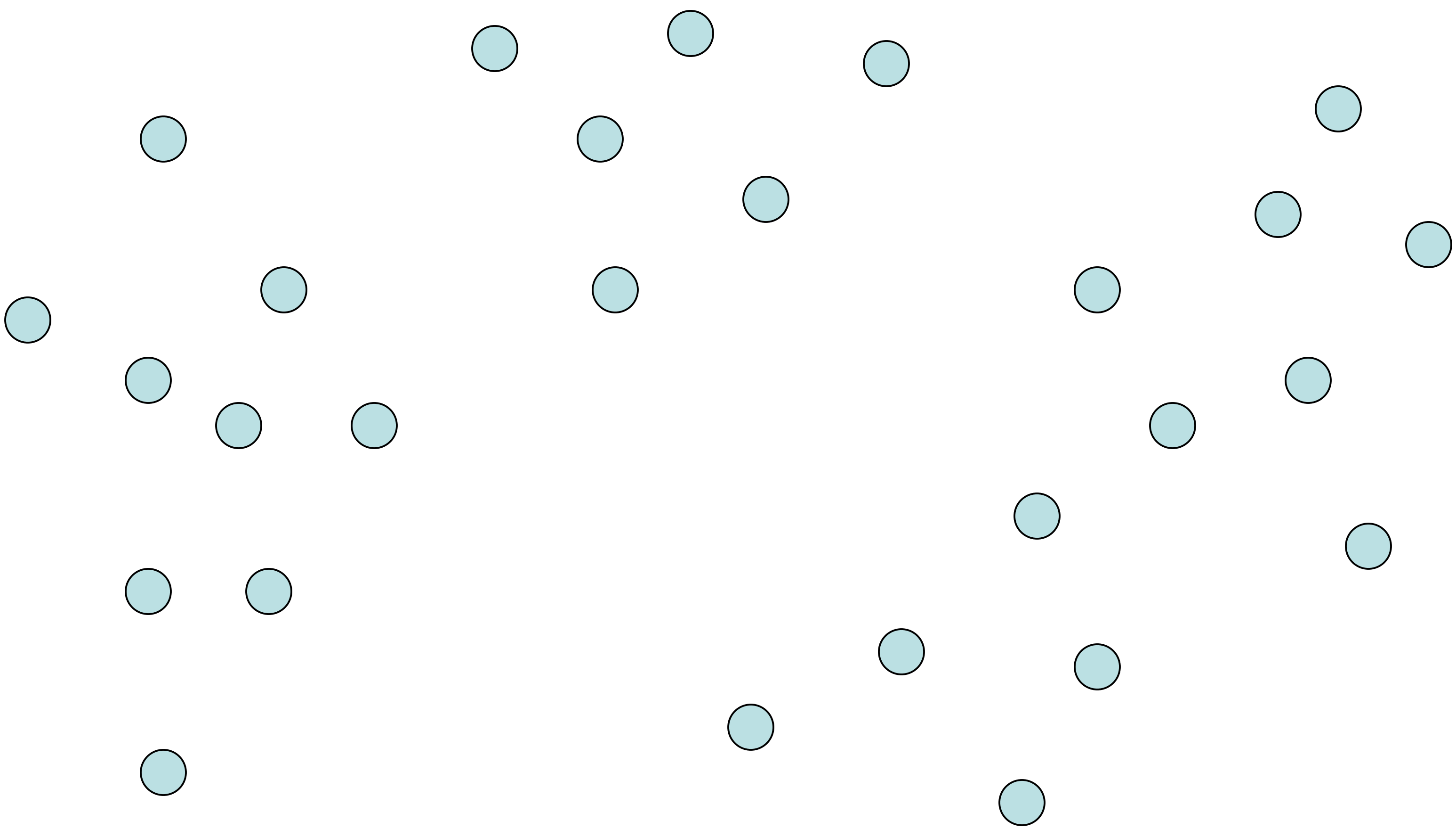
Divide into 2 clusters



Divide into 3 clusters



Divide into 4 clusters



Distance Clustering Algorithm

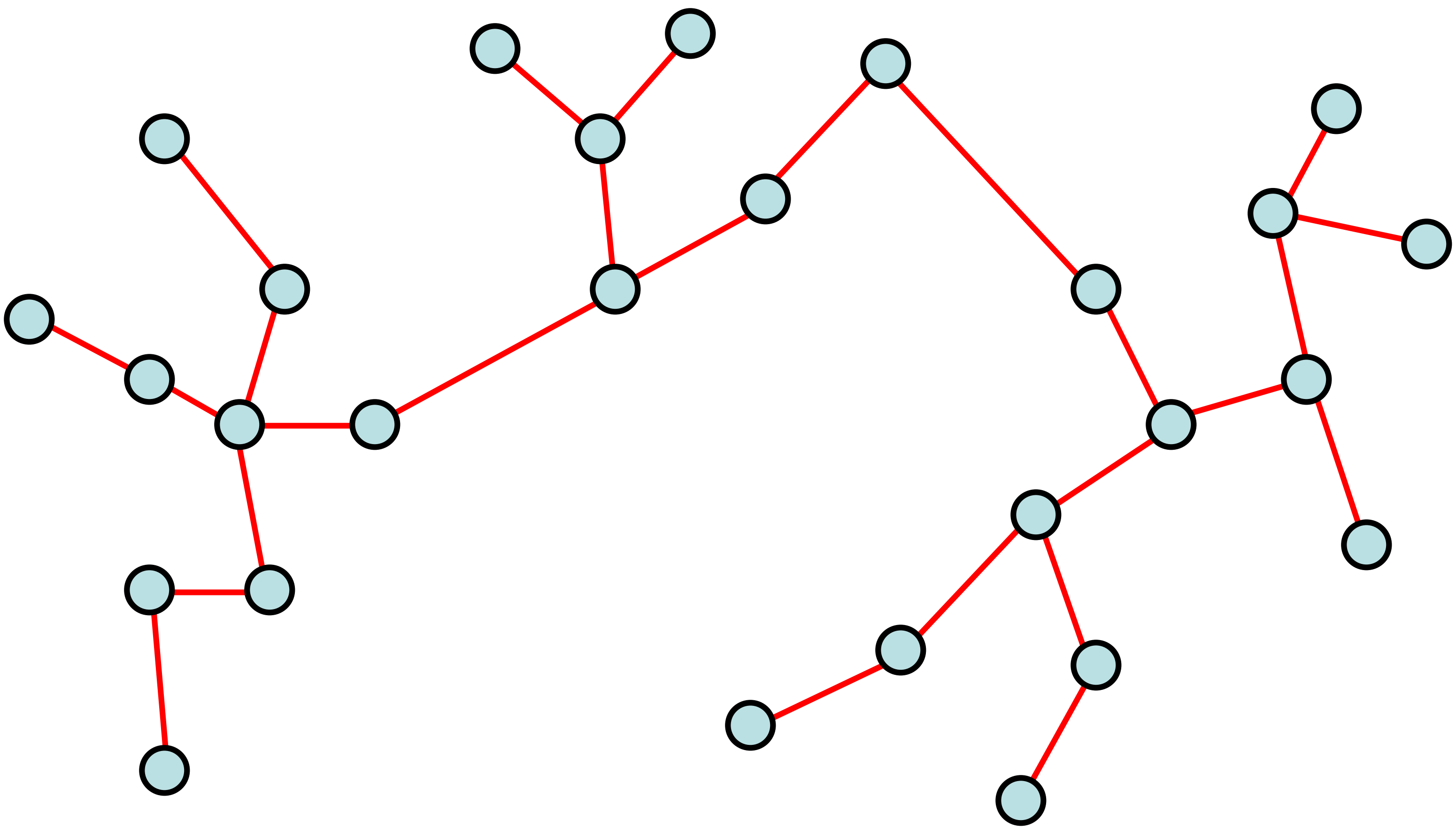
Let $C = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$; $T = \{\ \}$

while $|C| > K$

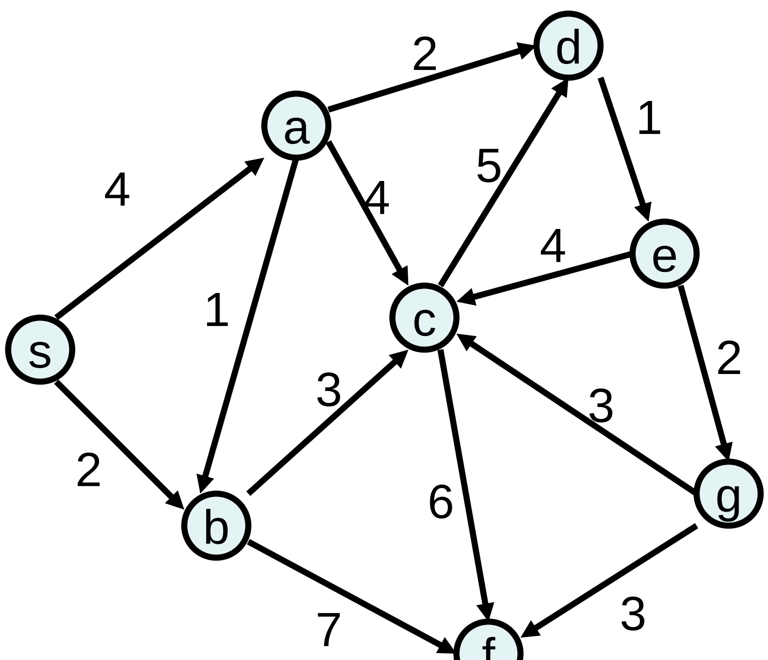
Let $e = (u, v)$ with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C

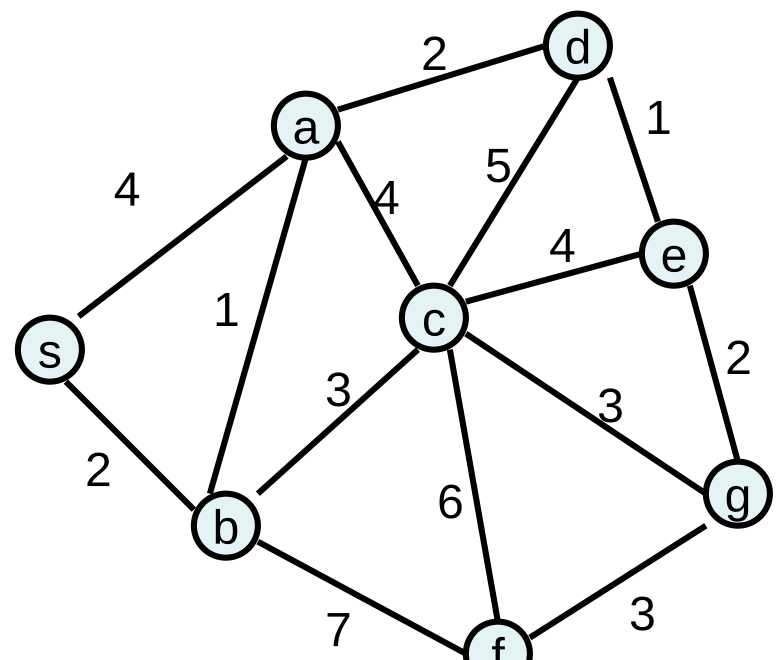
Replace C_i and C_j by $C_i \cup C_j$

K-clustering



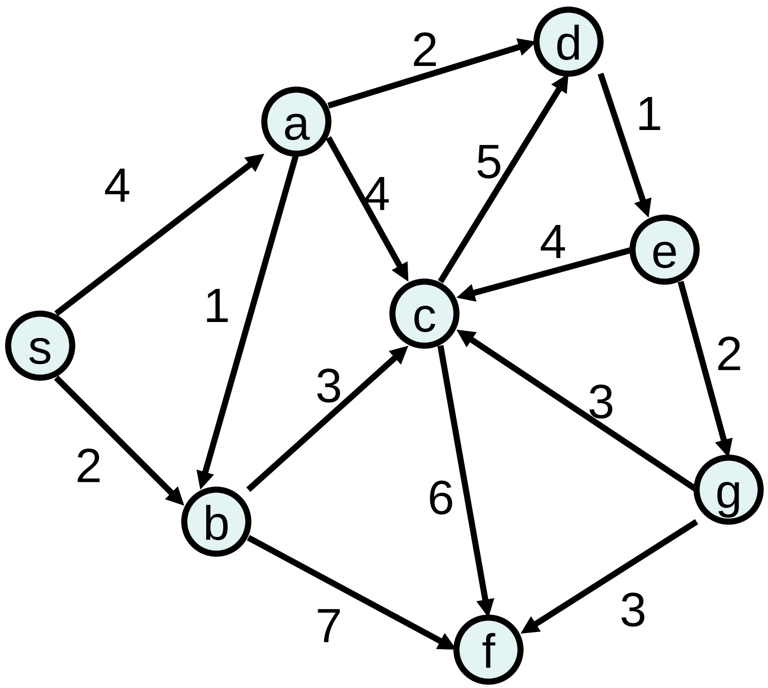
Shortest paths in directed graphs vs undirected graphs

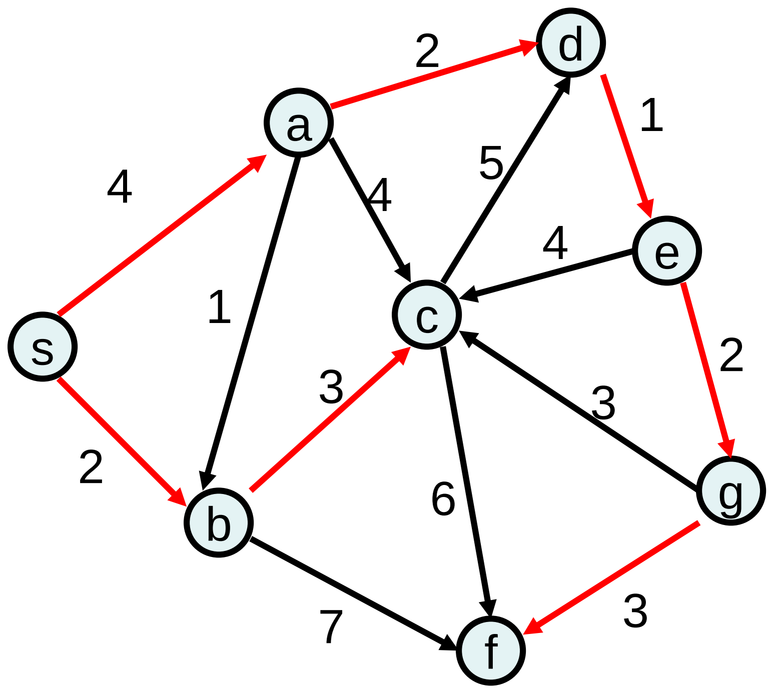




What about the minimum spanning tree of a directed graph?

- Must specify the root r
- Branching: Out tree with root r

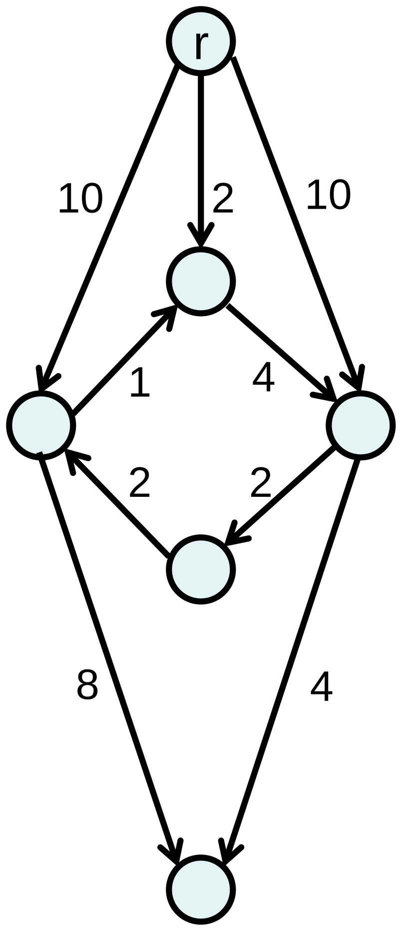


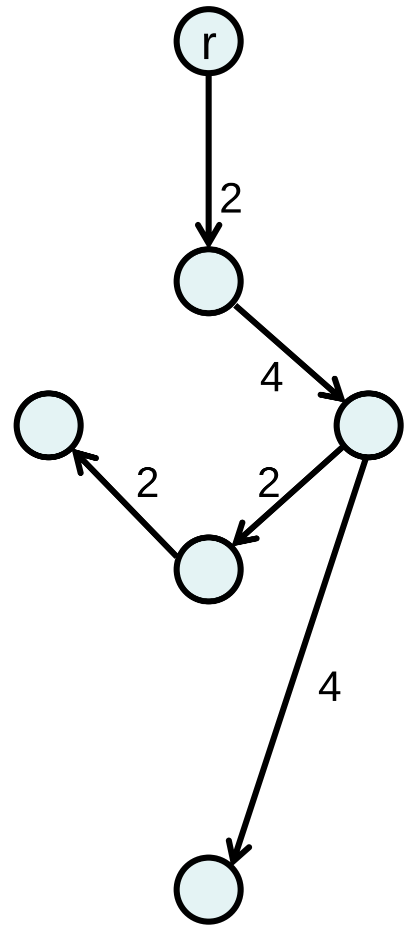


Assume all vertices reachable from r

Also called an arborescence

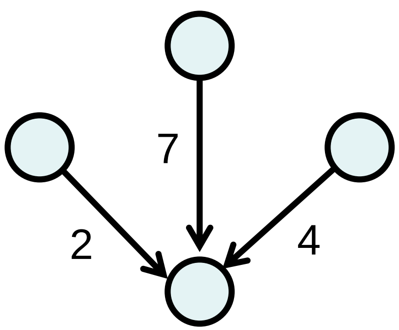
Finding a minimum branching

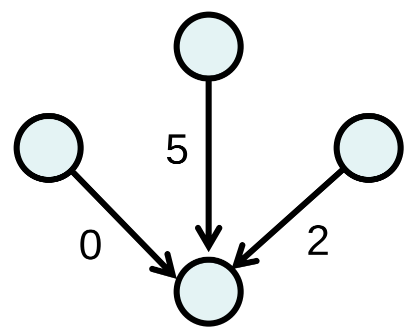




Finding a minimum branching

- Remove all edges going into r
- Normalize the edge weights, so the minimum weight edge coming into each vertex has weight zero



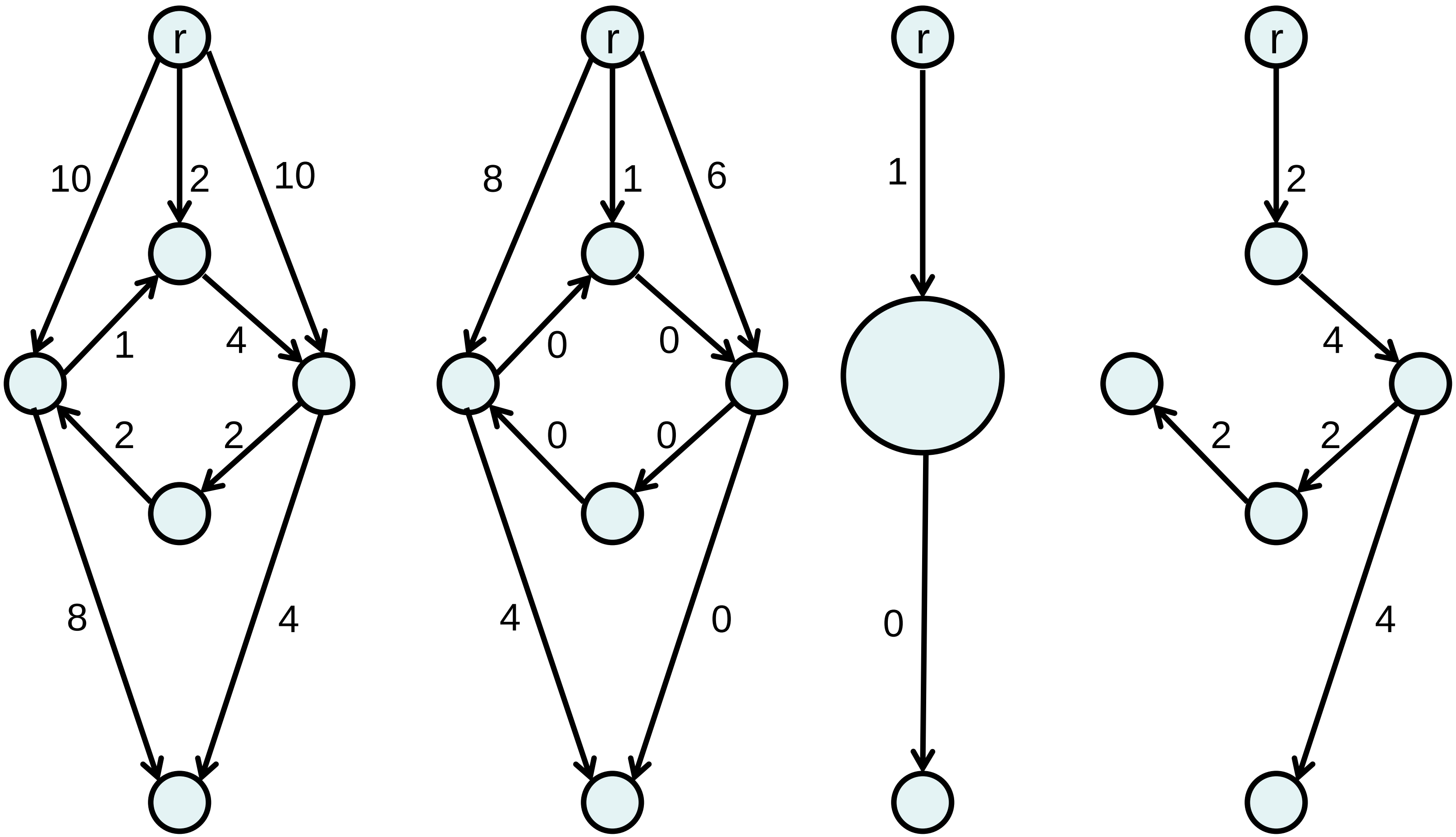


This does not change the edges of the minimum branching

Finding a minimum branching

- Consider the graph that consists of the minimum cost edge coming in to each vertex
  - If this graph is a branching, then it is the minimum cost branching
  - Otherwise, the graph contains one or more cycles
    - Collapse the cycles in the original graph to super vertices
    - Reweight the graph and repeat the process

Finding a minimum branching



Correctness Proof

Lemma 4.38 Let C be a cycle in G consisting of edges of cost 0 with r not in C. There is an optimal branching rooted at r that has exactly one edge entering C.

- The lemma justifies using the edges of the cycle in the branching
- An induction argument is used to cover the multiple levels of compressing cycles