CSE 421

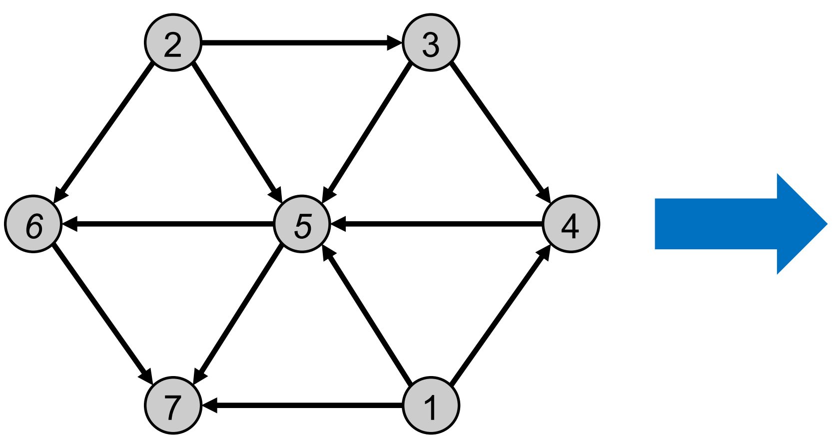
Topological Ordering / Greedy

Shayan Oveis Gharan

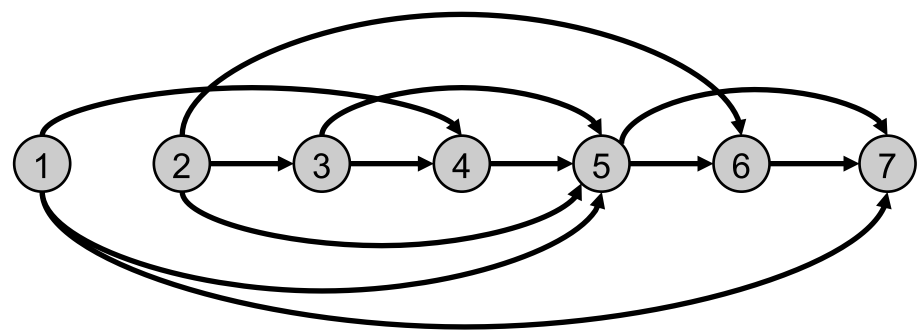
Directed Acyclic Graphs (DAG)

A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

Def: A topological order of a directed graph G = (V, E) is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



a DAG



a topological ordering of that DAG– all edges left-to-right

DAGs: A Sufficient Condition

Lemma: If G has a topological order, then G is a DAG.

Pf. (by contradiction)

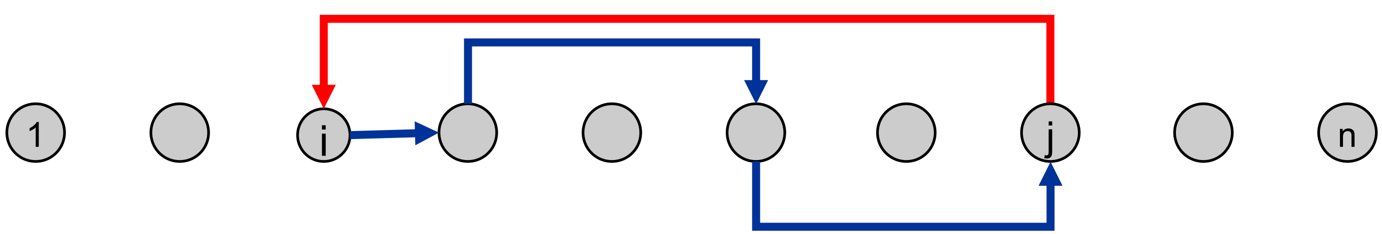
Suppose that G has a topological order $1, 2, \ldots, n$ and that G also has a directed cycle C.

Let i be the lowest-indexed node in C, and let j be the node just before i; thus (j, i) is an (directed) edge.

By our choice of i, we have $i < j$.

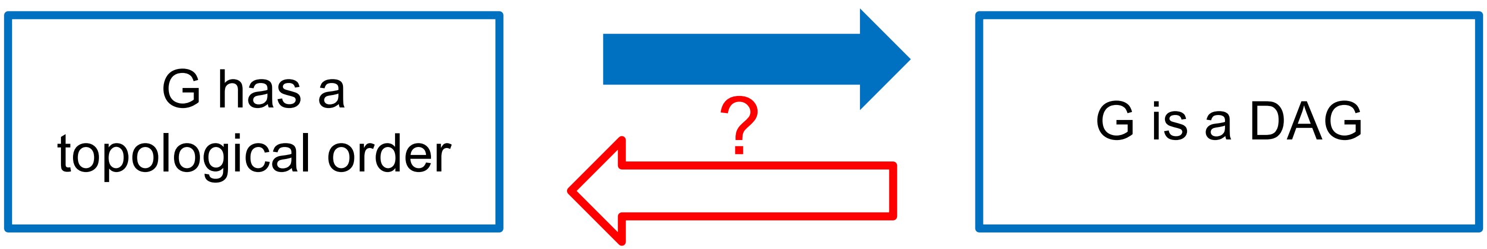
Have you seenthis idea before?
Yes! In analyzing Man-optimal stable matching

On the other hand, since (j, i) is an edge and $1, \ldots, n$ is a topological order, we must have $j < i$, a contradiction

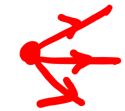


the supposed topological order: 1,2,…,n

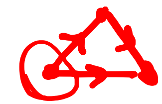
DAGs: A Sufficient Condition



Every DAG has a source node



Lemma: If G is a DAG, then G has a node with no incoming edges (i.e., a source).



Pf. (by contradiction)

Suppose that G is a DAG and and it has no source

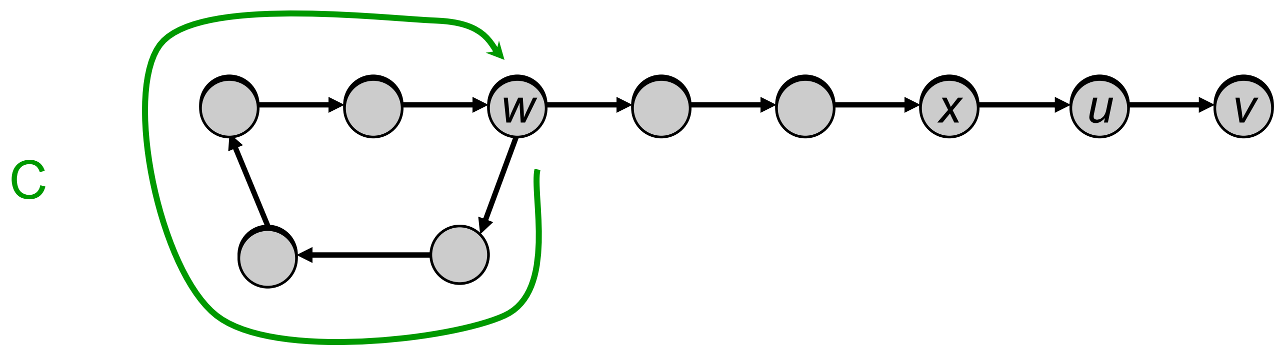
Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.

Then, since u has at least one incoming edge (x, u), we can walk backward to x.

Is this similar to a previous proof?

Repeat until we visit a node, say w, twice.

Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.



DAG => Topological Order

Lemma: If G is a DAG, then G has a topological order

Pf. (by induction on n)
Base case: true if n = 1.

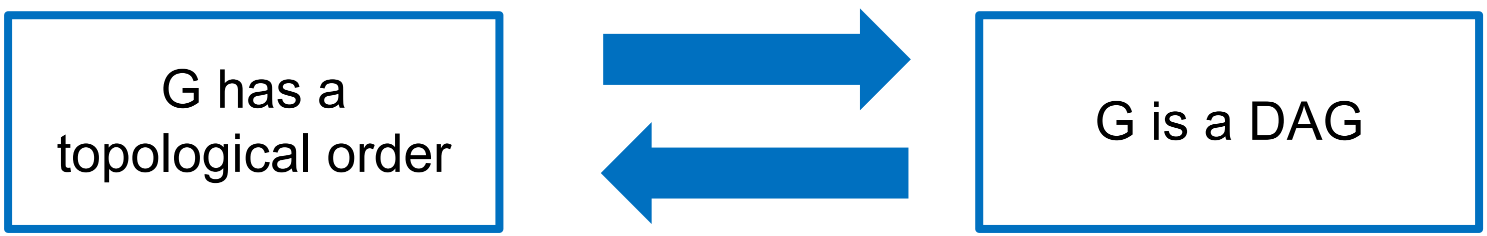
IH: Every DAG with n-1 vertices has a topological ordering.

IS: Given DAG with $n > 1$ nodes, find a source node v.
$G - \{ v \}$ is a DAG, since deleting v cannot create cycles.

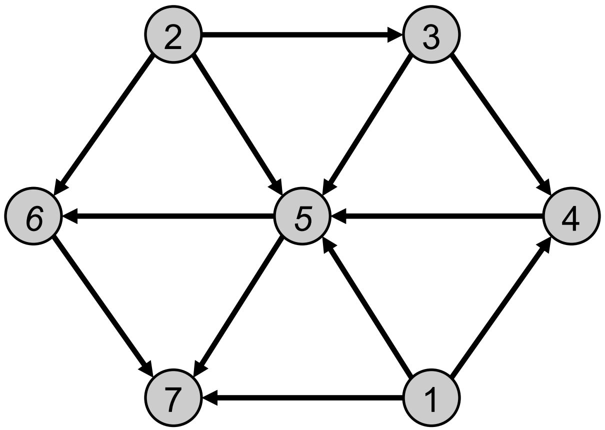
Reminder: Always remove vertices/edges to use IH

By IH, $G - \{ v \}$ has a topological ordering.
Place v first in topological ordering; then append nodes of G - { v } in topological order. This is valid since v has no incoming edges.

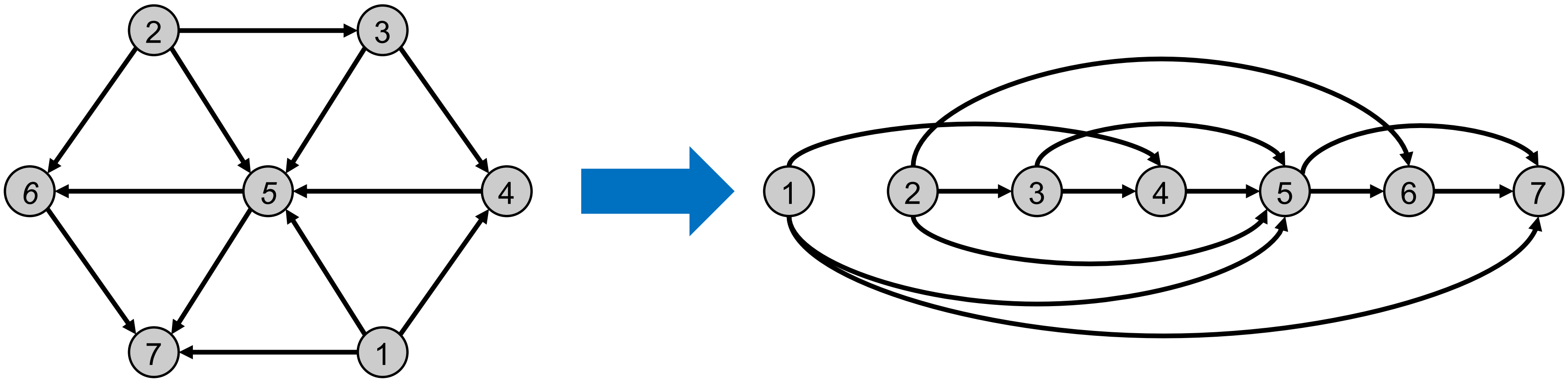
A Characterization of DAGs



Topological Order Algorithm: Example



Topological Order Algorithm: Example



Topological order: 1, 2, 3, 4, 5, 6, 7

Topological Sorting Algorithm

Maintain the following:

count[w] = (remaining) number of incoming edges to node w

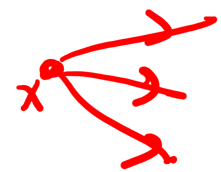
S = set of (remaining) nodes with no incoming edges

Initialization:

count[w] = 0 for all w

count[w]++ for all edges (v,w) — O(m + n)

S = S ∪ {w} for all w with count[w]=0



Main loop:

while S not empty

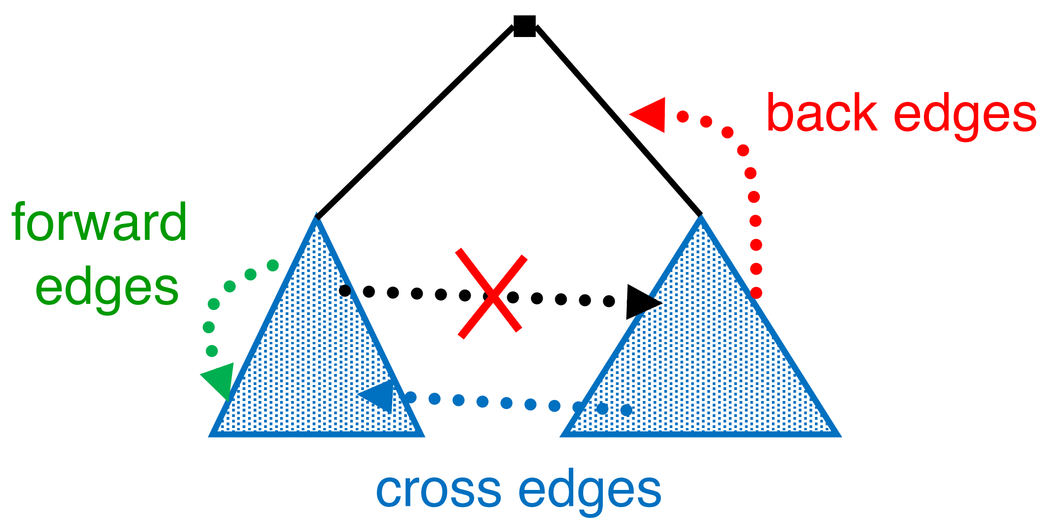
- remove some v from S
- make v next in topo order — O(1) per node
- for all edges from v to some w — O(1) per edge
  - –decrement count[w]
  - –add w to S if count[w] hits 0

Correctness: clear, I hope

Time: O(m + n) (assuming edge-list representation of graph)

DFS on Directed Graphs

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s
- Every cycle contains a back edge in the DFS tree



Summary

- Graphs: abstract relationships among pairs of objects
- Terminology: node/vertex/vertices, edges, paths, multi-edges, self-loops, connected
- Representation: Adjacency list, adjacency matrix
- Nodes vs Edges: $m = O(n^2)$, often less
- BFS: Layers, queue, shortest paths, all edges go to same or adjacent layer
- DFS: recursion/stack; all edges ancestor/descendant
- Algorithms: Connected Comp, bipartiteness, topological sort

Greedy Algorithms

Greedy Strategy

Goal: Given currency denominations: 1, 5, 10, 25, 100, give change to customer using *fewest* number of coins.

Ex: 34¢.

Cashier's algorithm: At each iteration, give the *largest* coin valued ≤ the amount to be paid.

Ex: $2.89.

Greedy is not always Optimal

Observation: Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

Greedy: 100, 34, 1, 1, 1, 1, 1, 1.

Optimal: 70, 70.

Lesson: Greedy is short-sighted. Always chooses the most attractive choice at the moment. But this may lead to a dead-end later.

Greedy Algorithms Outline

Pros

- Intuitive
- Often simple to design (and to implement)
- Often fast

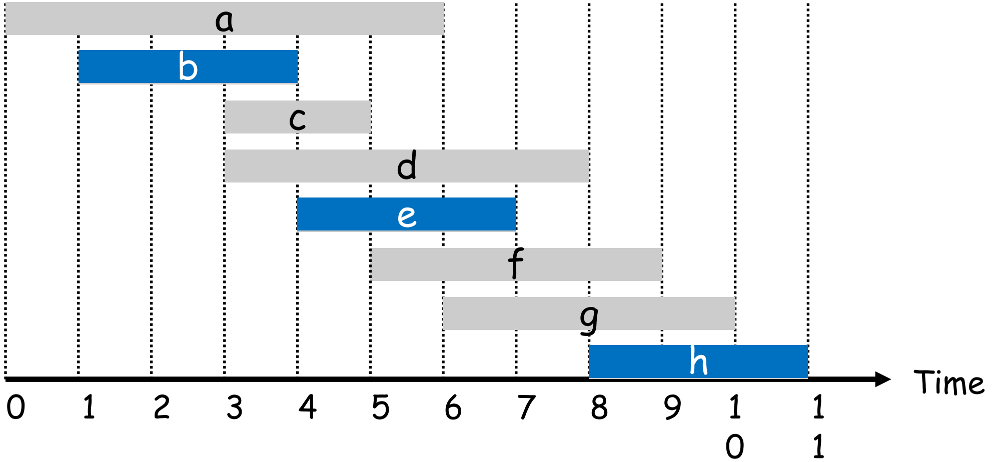
Cons

- Often incorrect!

Proof techniques:

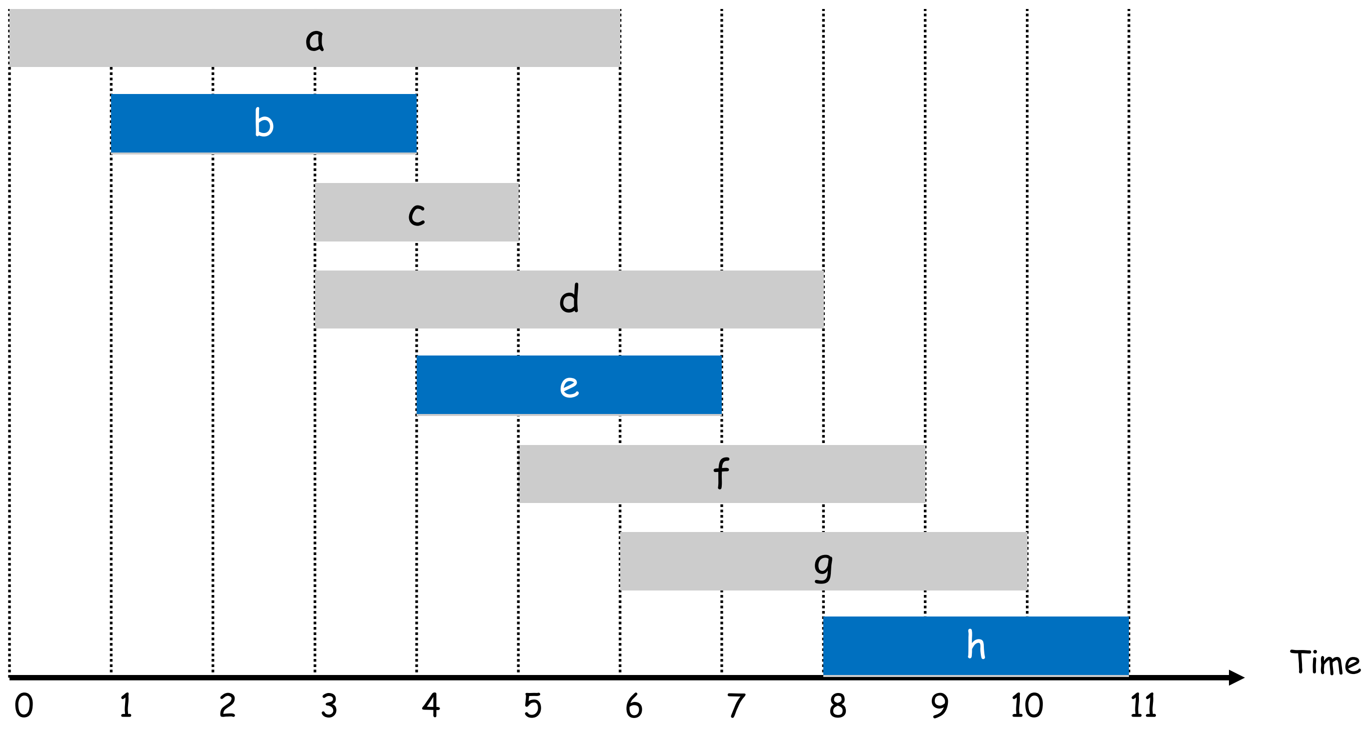
- Stay ahead
- Structural
- Exchange arguments

Interval Scheduling



Interval Scheduling

- Job j starts at $s(j)$ and finishes at $f(j)$.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Greedy Strategy

Sort the jobs in some order. Go over the jobs and take as much as possible provided it is compatible with the jobs already taken.

Main question:

- What order?
- Does it give the Optimum answer?
- Why?

Possible Approaches for Inter Sched

Sort the jobs in some order. Go over the jobs and take as much as possible provided it is compatible with the jobs already taken.

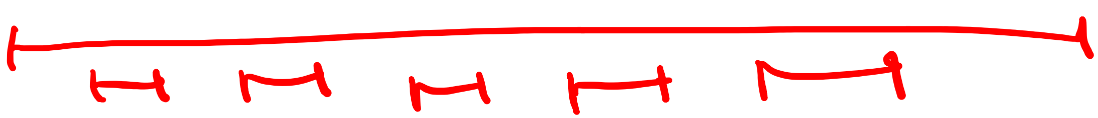
[Earliest start time] Consider jobs in ascending order of start time s_j.

works

[Earliest finish time] Consider jobs in ascending order of finish time f_j.

[Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.

[Fewest conflicts] For each job, count the number of conflicting jobs c_j. Schedule in ascending order of conflicts c_j.



Greedy Alg: Earliest Finish Time

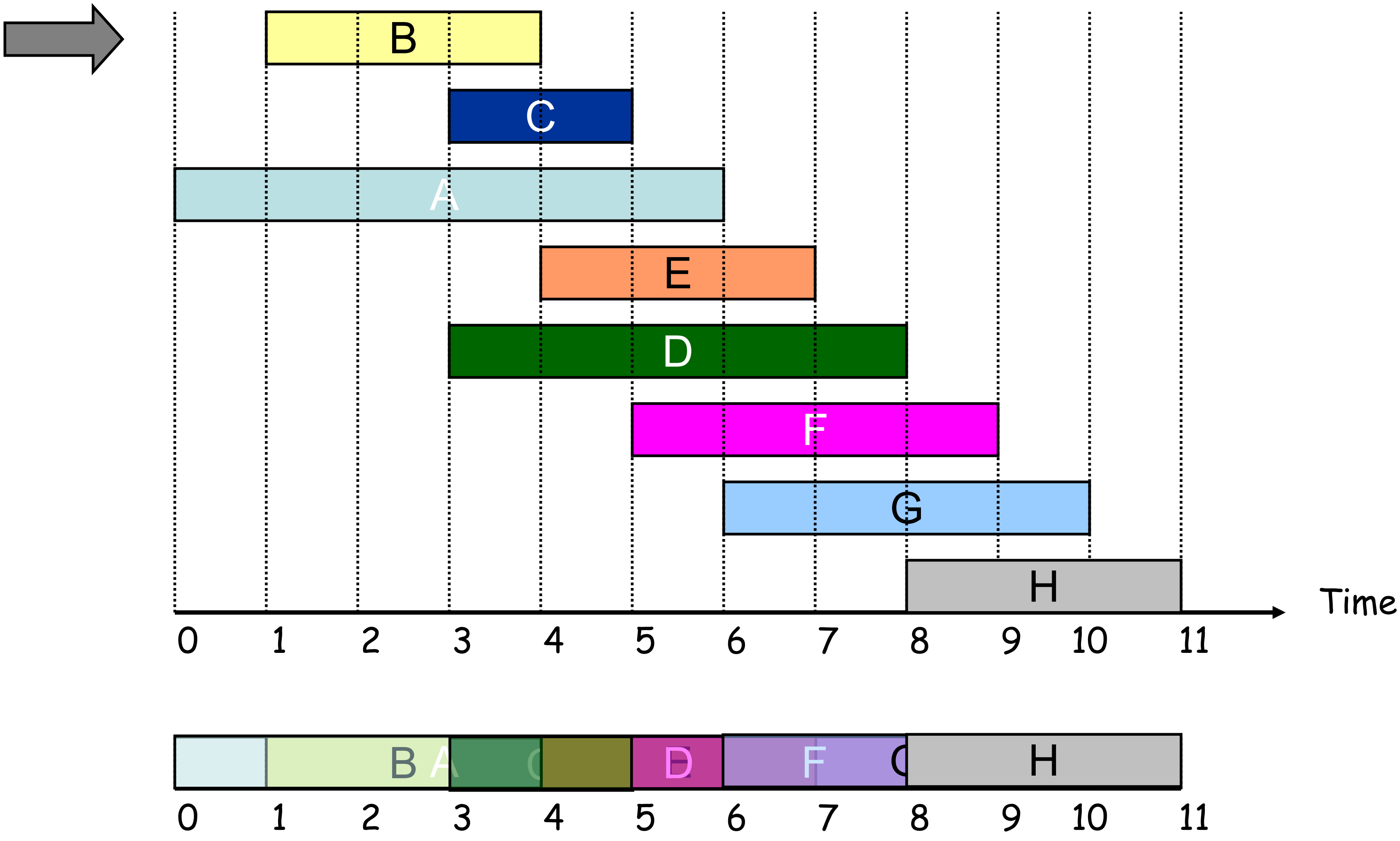
Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that f(1) ≤ f(2) ≤ ... ≤ f(n).
A ← ∅
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

Implementation. O(n log n).

- Remember job j* that was added last to A.
- Job j is compatible with A if $s(j) \geq f(j^*)$.

Greedy Alg: Example



Correctness

Theorem: Greedy algorithm is optimal.

Pf: (technique: "Greedy stays ahead")

Let $i_1, i_2, \ldots i_k$ be jobs picked by greedy, $j_1, j_2, \ldots j_m$ those in some optimal solution in order.

We show $f(i_r) \leq f(j_r)$ for all r, by induction on r.

Base Case: i_1 chosen to have min finish time, so $f(i_1) \leq f(j_1)$.

IH: $f(i_r) \leq f(j_r)$ for some r

IS: Since $f(i_r) \leq f(j_r) \leq s(j_{r+1})$, j_{r+1} is among the candidates considered by greedy when it picked i_{r+1}, & it picks min finish, so $f(i_{r+1}) \leq f(j_{r+1})$

Observe that we must have $k \geq m$, else j_{k+1} is among (nonempty) set of candidates for i_{k+1}