Lem: If G has a topological ordering then G is a DAG.

Pf. [Pf by contradiction]. Assume G has a cycle and has a topological order.

topolog ordering

First way

$C = a_1 a_2 \cdots a_k$

$a_1 \to a_2$ implies $a_2 > a_1$

$a_2 \to a_3 \Rightarrow a_3 > a_2$

$\vdots$

$a_{k-1} \to a_k \Rightarrow a_k > a_{k-1}$

$a_k \to a_1 \Rightarrow a_1 > a_k$

Contradiction!

Second way

Pick smallest index vertex in C a_i. Then, $a_{i-1} \to a_i$

BC a_i has smaller index $a_i < a_{i-1}$

By topolog ordering $a_{i-1} < a_i$

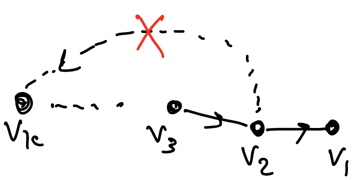
contradiction!

Lem, Every DAG has a source node.

Pf

Start with v_1 either a source node ✓

or v_2 points to v_1



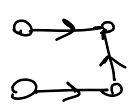
keep doing this: Given $v_k \to v_k$ source ✓

$\to$ one of $v_1 \cdots v_{k-1}$ points to v_k but that gives a cycle Not possible

$\to$ call the new node pointing to v_k, v_{k+1}

This has to stop BC G is finite

So we get a source node.



$\square$

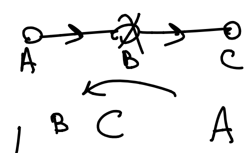
Lem: If G is a DAG it has a topological ordering.

Pf. Induction

Base Case: DAG with one node • ✓

IH: Every DAG with $n-1$ nodes has a topological order

IS: Given a DAG G with n nodes: Construct topo order.

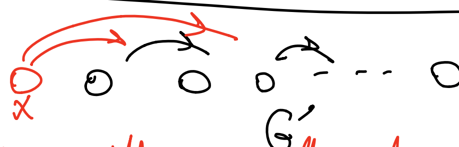


An arbitrary removal fails.

Remove a source node x.

Remaining graph G' is a DAG.

By IH G' has a top order



Add x to be first vertex. You will get all edges pointing right. $\square$