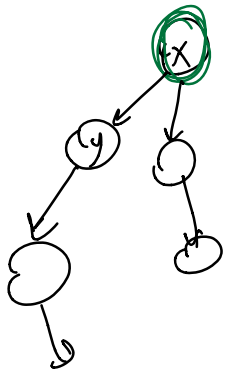
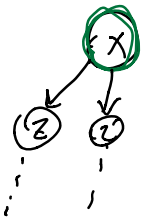


OBS. Every vertex z which is discovered while x is still in stack is a descendant of x.

Lem: All non-tree edges are ancestor/descendant.

Pf: Fix some edge $\{x,y\}$.

Supp x is discovered first.



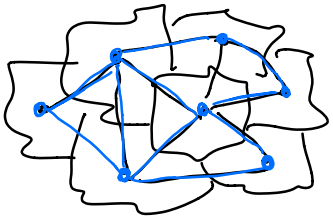
I claim, before x is fully discovered, we discover y.

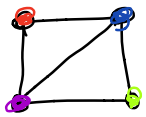
BC before finishing up with x, we will look at all neighbors of x, and discover them if not yet discovered.

y is a neighbor and it will be discovered.

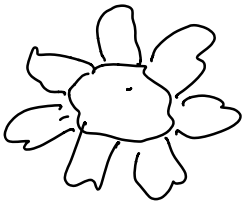
$\Rightarrow$ by OBS y is a descendant of x. ∎

Thm. Vertices of all planar graph can be colored with 6 colors.





Claim: Every planar graph has a vertex of $\deg \leq 5$.



Pf. [Fact: For all plan graph $3n-4 \geq m$]

$\sum_v \deg(v) = 2m \leq 2(3n-4) = 6n-8$.

If $\deg(v) \geq 6$ for all $v \Rightarrow \sum_v \deg(v) \geq 6n$ contradiction ∎

Pf of Thm.

Goal: Evy plm with n vertices can be colored with 6 colors.

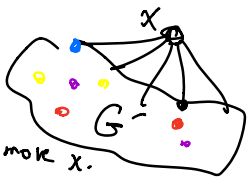
Base Case: n=1 ✓

IH: Evy planar graph with n-1 vertice can be colored with 6 colors.

IS. Given planar G with n vertices. We want to color G with 6 colors.

G has vertex x of $d_G(x) \leq 5$.

Remove x and call G' (the remaing).



Fact: G' is plmr BC G was plmr and we remove x.

By IH color G' with 6 colors.

Just enough to color x.

x has 5 neighbors $\Rightarrow \exists$ color not used on neighbors of x.

Color x with that ∎

Lem Given directed graph G with a topological order G must be a PAG.

Pf.