CSE 421: Introduction to Algorithms

Induction - Graphs

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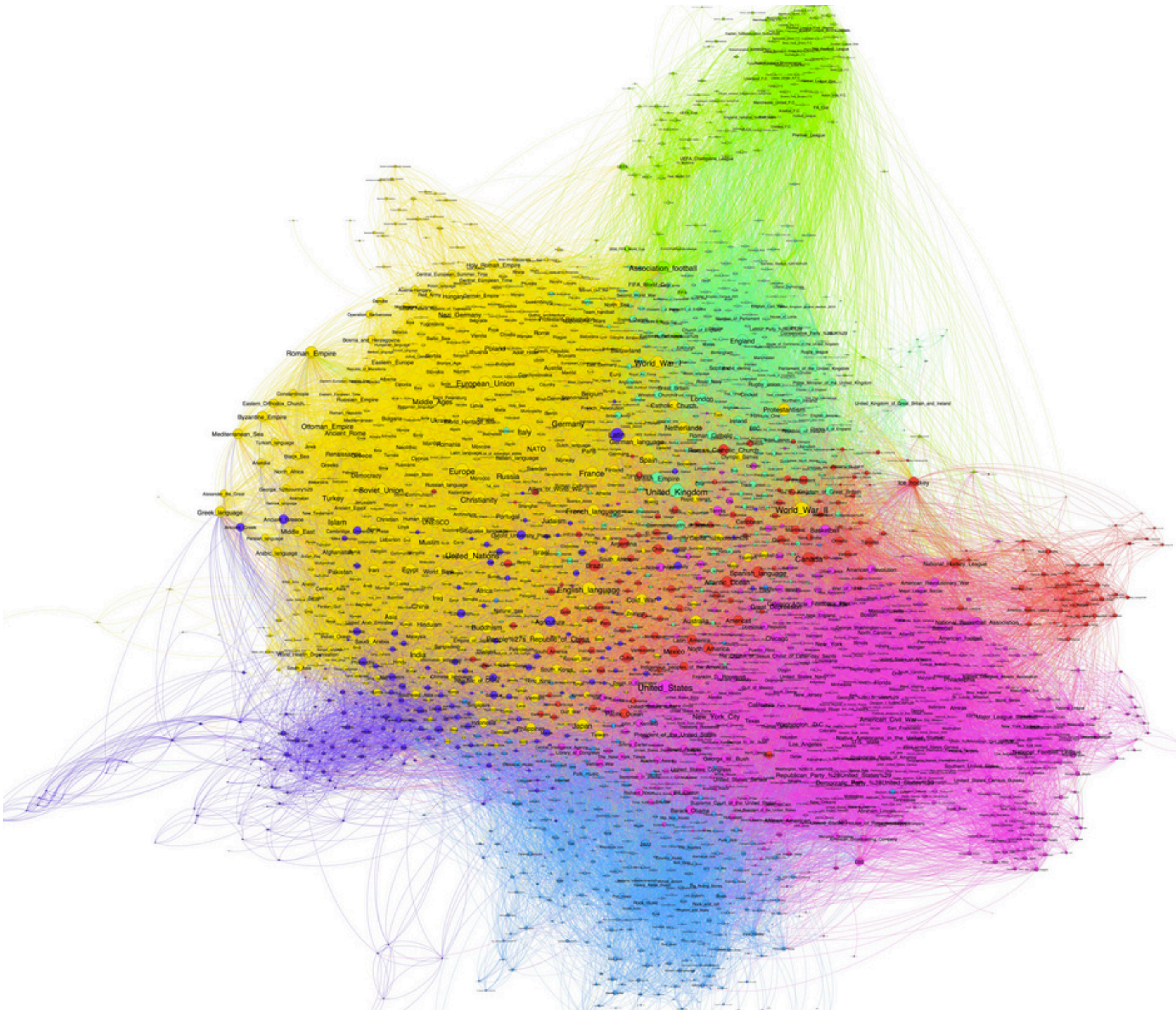
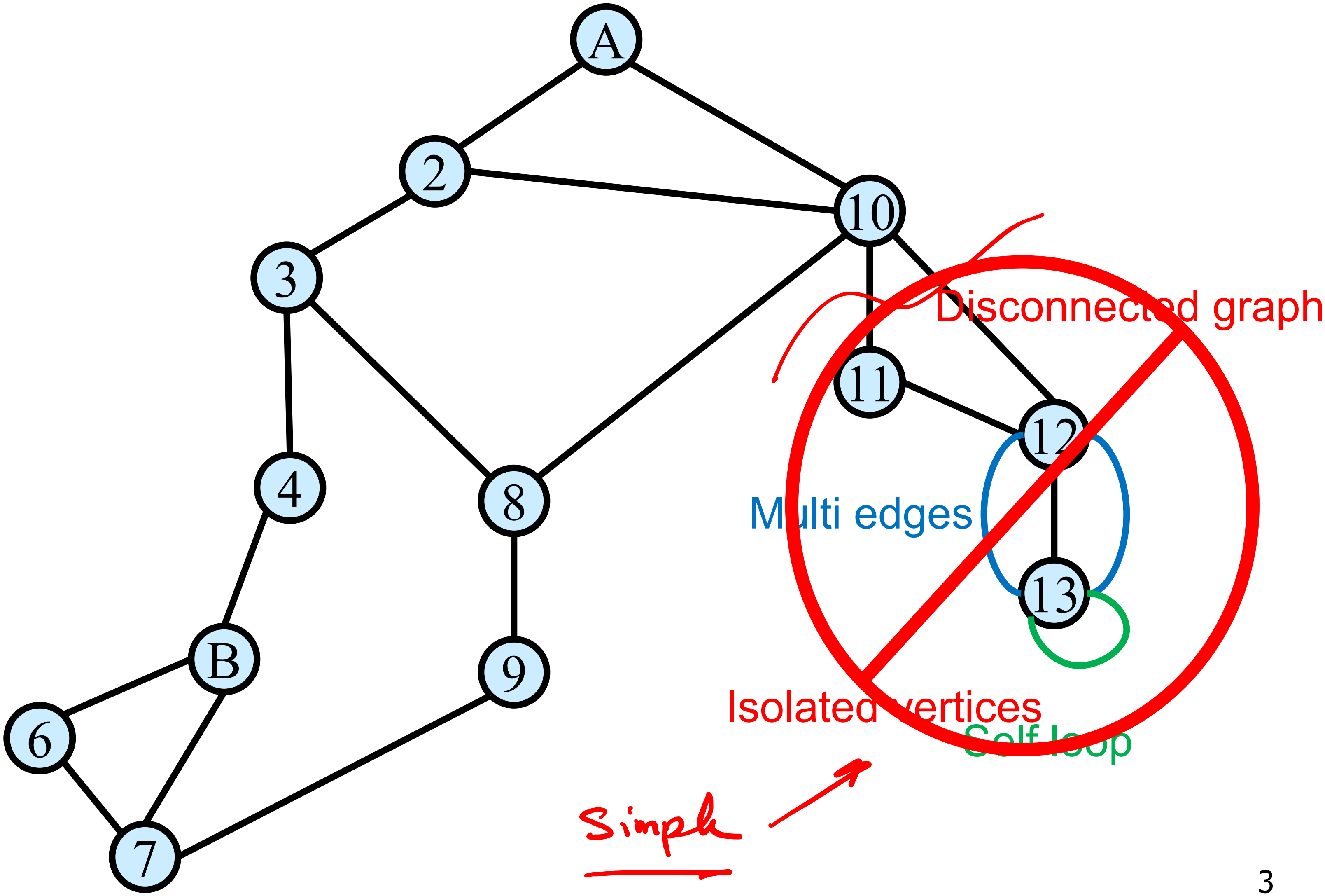
Graphs



Association_football
Roman_Empire
European_Union
World_War_I
England
London
Germany
Italy
Spain
France
Europe
Russia
United_Kingdom
World_War_II
Canada
English_language
United_States

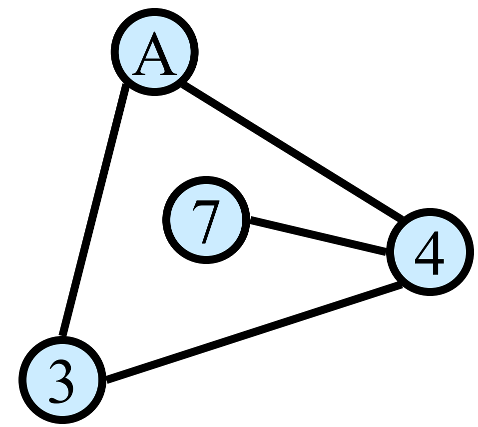
Undirected Graphs G=(V,E)

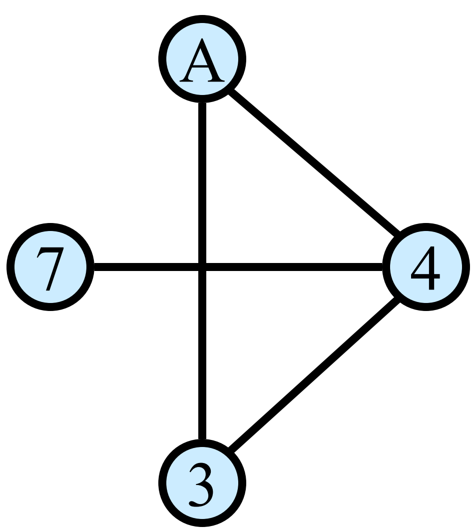


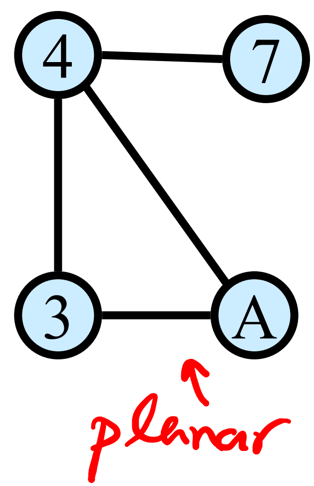
Graphs don't Live in Flat Land

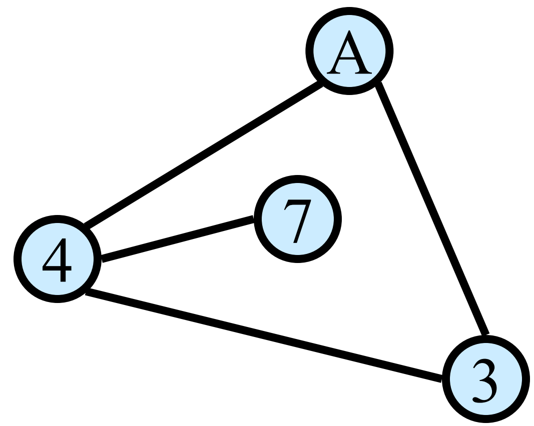
Geometrical drawing is mentally convenient, but mathematically irrelevant:

4 drawings of a single graph:

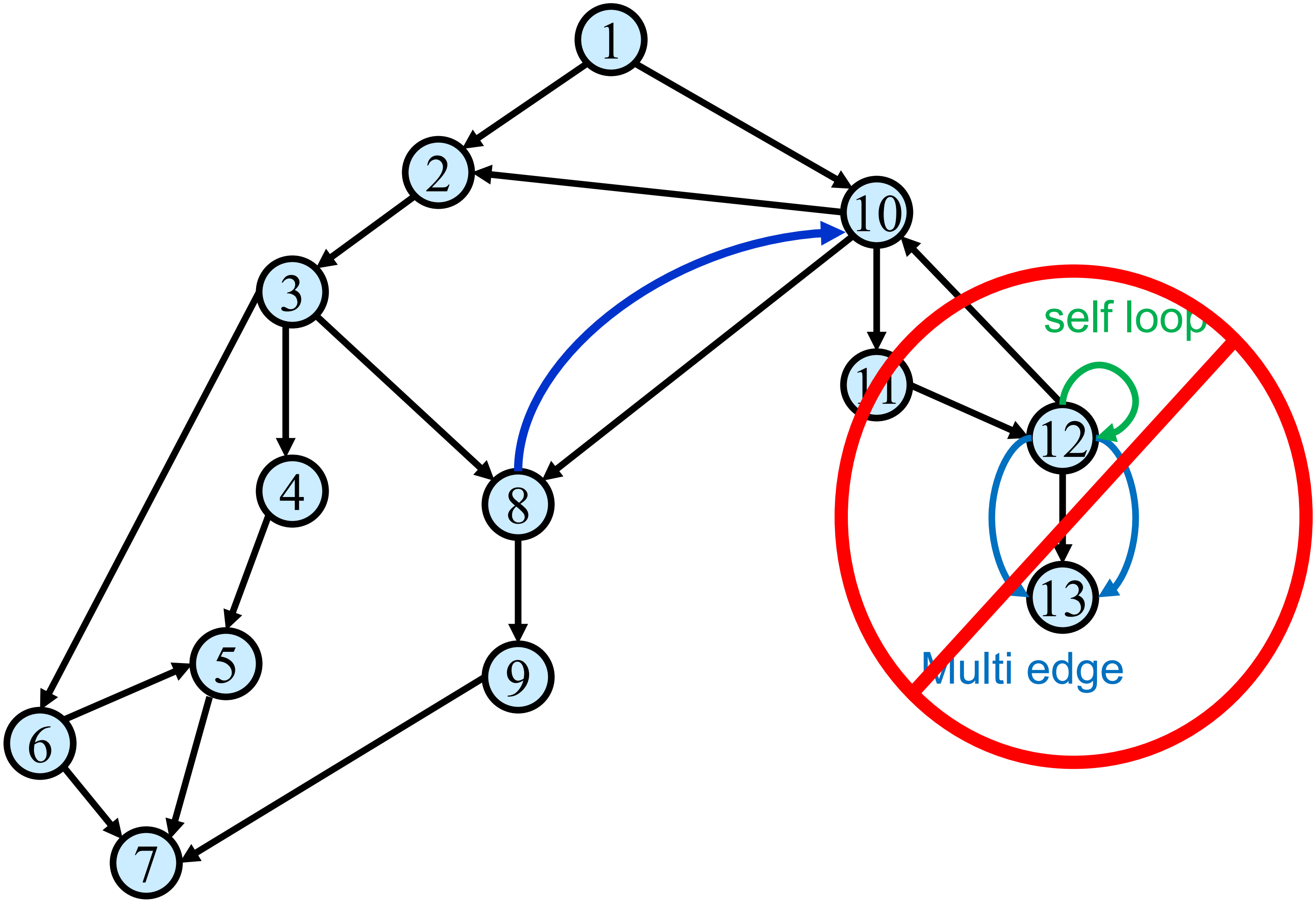






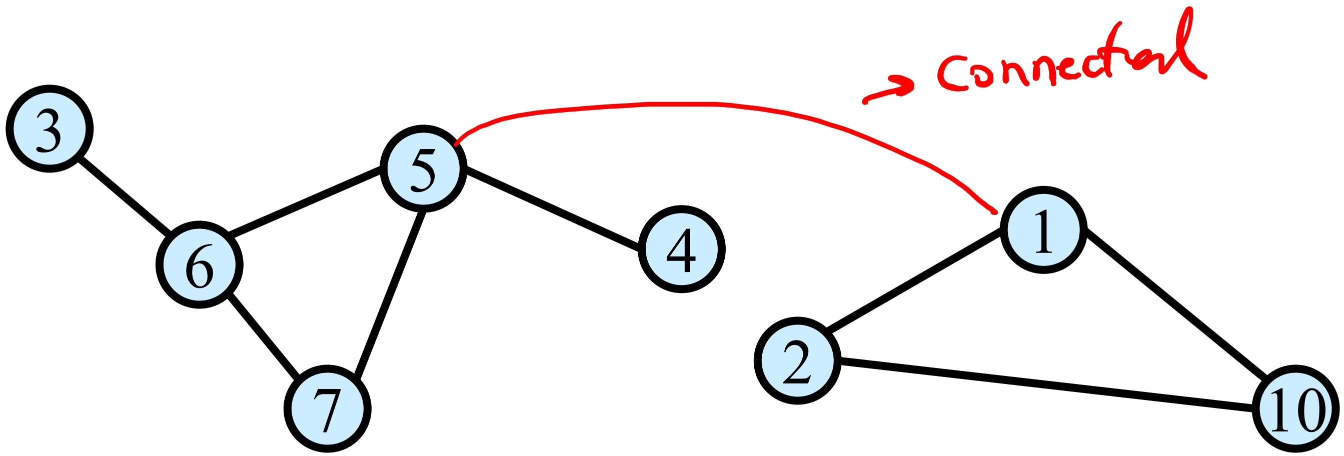


Directed Graphs



Terminology

- Degree of a vertex: # edges that touch that vertex

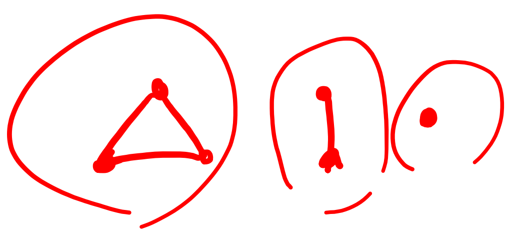


deg(6)=3

deg(10)=2

- Connected: Graph is connected if there is a path between every two vertices

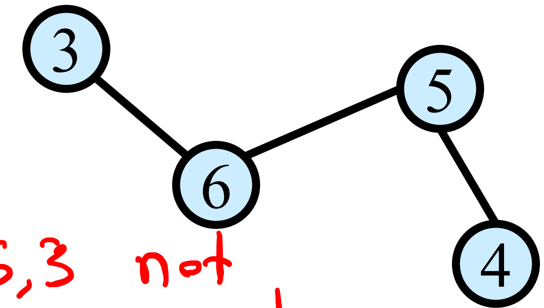
- Connected component: Maximal set of connected vertices



3 conn

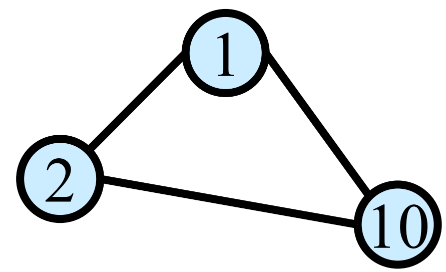
Terminology (cont'd)

- Path: A sequence of distinct vertices s.t. each vertex is connected to the next vertex with an edge

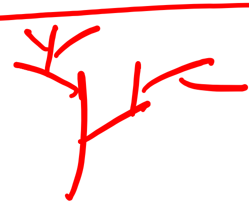


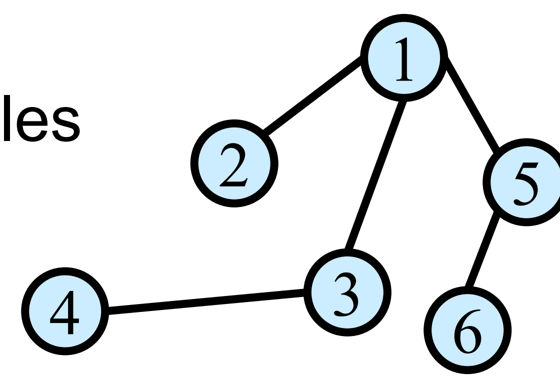
3,6,5,3 not a path

- Cycle: Path of length > 2 that has the same start and end



- Tree: A connected graph with no cycles

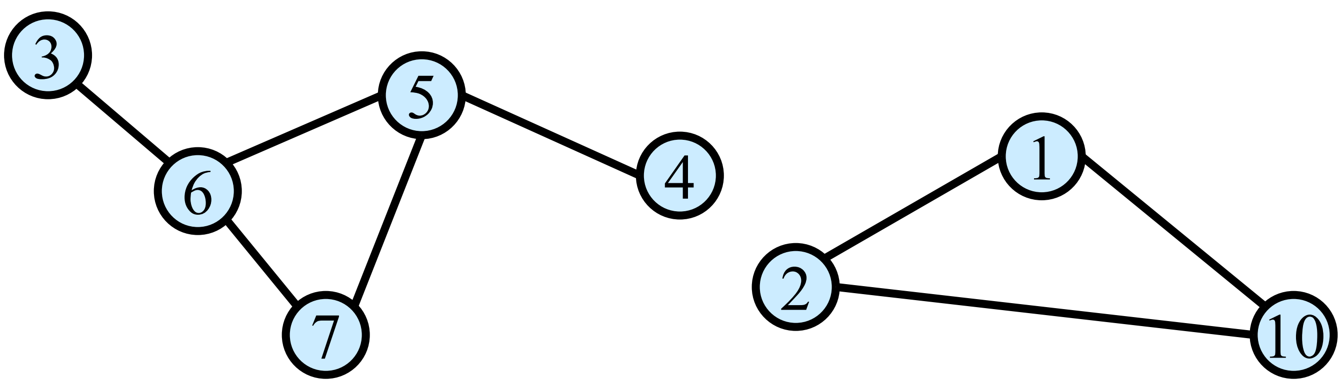




Degree Sum

Claim: In any undirected graph, the number of edges is equal to $(1/2)\sum_{\text{vertex } v} \deg(v)$

Pf: $\sum_{\text{vertex } v} \deg(v)$ counts every edge of the graph exactly twice; once from each end of the edge.



|E|=8

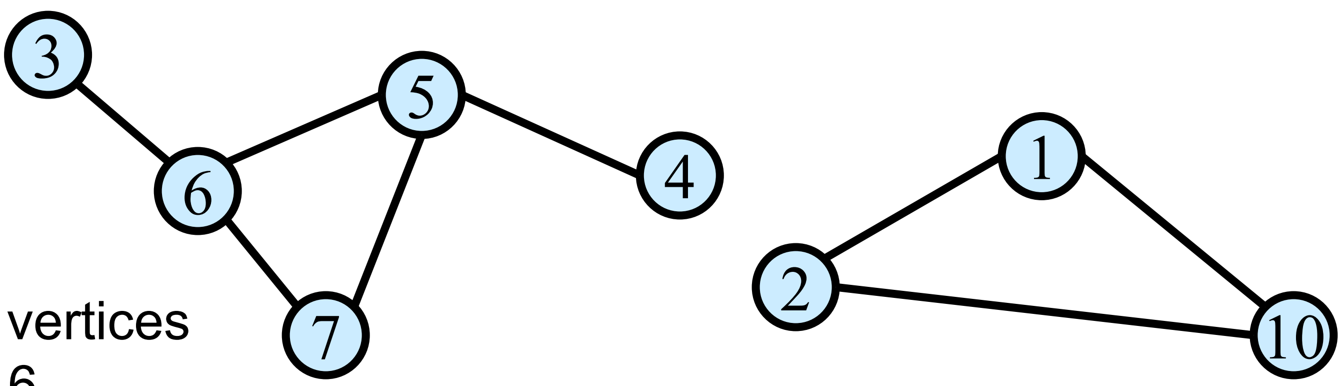
$$\sum_{\text{vertex } v} \deg(v) = 2+2+1+1+3+2+3+2 = 16$$

Odd Degree Vertices

Claim: In any undirected graph, the number of odd degree vertices is even

Pf: In previous claim we showed sum of all vertex degrees is even. So there must be even number of odd degree vertices, because sum of odd number of odd numbers is odd.

sum of odd number of odd numbers is odd.



4 odd degree vertices
3, 4, 5, 6

Degree 1 vertices

Claim: If G has no cycle, then it has a vertex of degree ≤ 1

(So, every tree has a leaf)

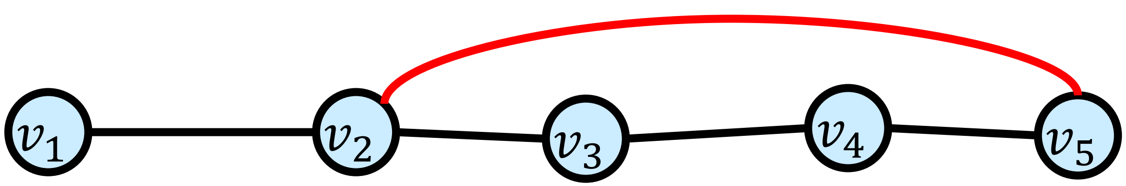
Pf: (By contradiction)

Suppose every vertex has degree ≥ 2.

Start from a vertex v_1 and follow a path, $v_1, \ldots, v_i$ when we are at v_i we choose the next vertex to be different from v_{i-1}. We can do so because $\deg(v_i) \geq 2$.

The first time that we see a repeated vertex ($v_j = v_i$) we get a cycle.

We always get a repeated vertex because G has finitely many vertices



Trees and Induction

Claim: Show that every tree with n vertices has n-1 edges.

Pf: By induction.

Base Case: n=1, the tree has no edge

IH: Suppose every tree with n-1 vertices has n-2 edges

IS: Let T be a tree with n vertices.

So, T has a vertex v of degree 1.

Remove v and the neighboring edge, and let T' be the new graph.

We claim T' is a tree: It has no cycle, and it must be connected.

So, T' has n-2 edges and T has n-1 edges.

#edges

Let $G = (V, E)$ be a graph with $n = |V|$ vertices and $m = |E|$ edges.

Claim: $0 \le m \le \binom{n}{2} = \frac{n(n-1)}{2} = O(n^2)$

Pf: Since every edge connects two distinct vertices (i.e., G has no loops)

and no two edges connect the same pair of vertices (i.e., G has no multi-edges)

It has at most $\binom{n}{2}$ edges.

Sparse Graphs

A graph is called sparse if $m \ll n^2$ and it is called dense otherwise.

Sparse graphs are very common in practice

- Friendships in social network
- Planar graphs
- Web braph

Q: Which is a better running time $O(n+m)$ vs $O(n^2)$?

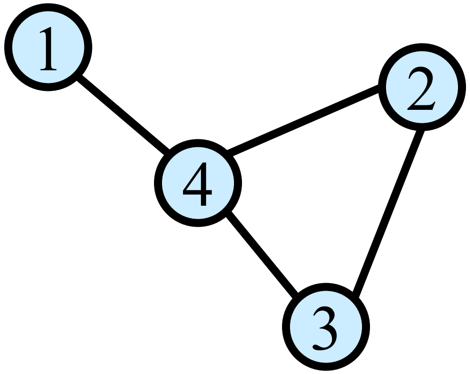
A: $O(n+m) = O(n^2)$, but $O(n+m)$ is usually much better.

Storing Graphs (Internally in ALG)

Vertex set $V = \{v_1, \dots, v_n\}$.

Adjacency Matrix: A

- For all, $i, j, A[i,j] = 1$ iff $(v_i, v_j) \in E$
- Storage: n^2 bits



	1	2	3	4
1	0	0	0	1
2	0	0	1	1
3	0	1	0	1
4	1	1	1	0

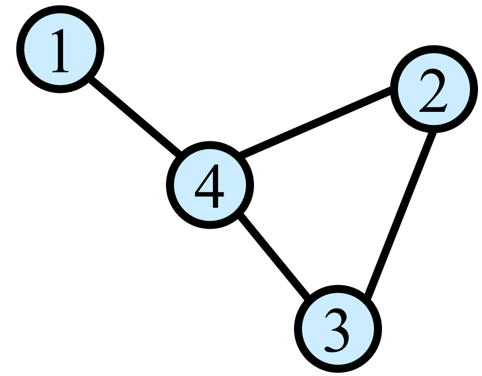
Advantage:

- $O(1)$ test for presence or absence of edges

Disadvantage:

- Inefficient for sparse graphs both in storage and edge-access

Storing Graphs (Internally in ALG)

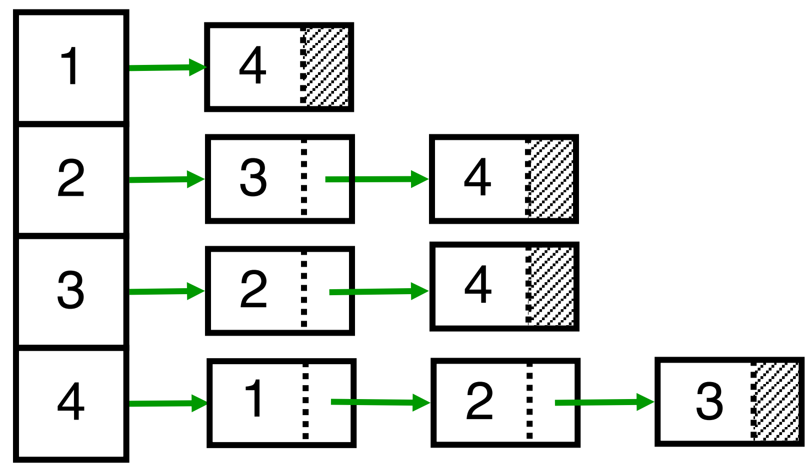


Adjacency List:

O(n+m) words

Advantage

- Compact for sparse
- Easily see all edges



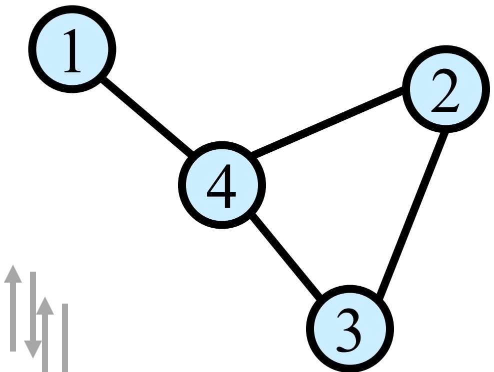
Disadvantage

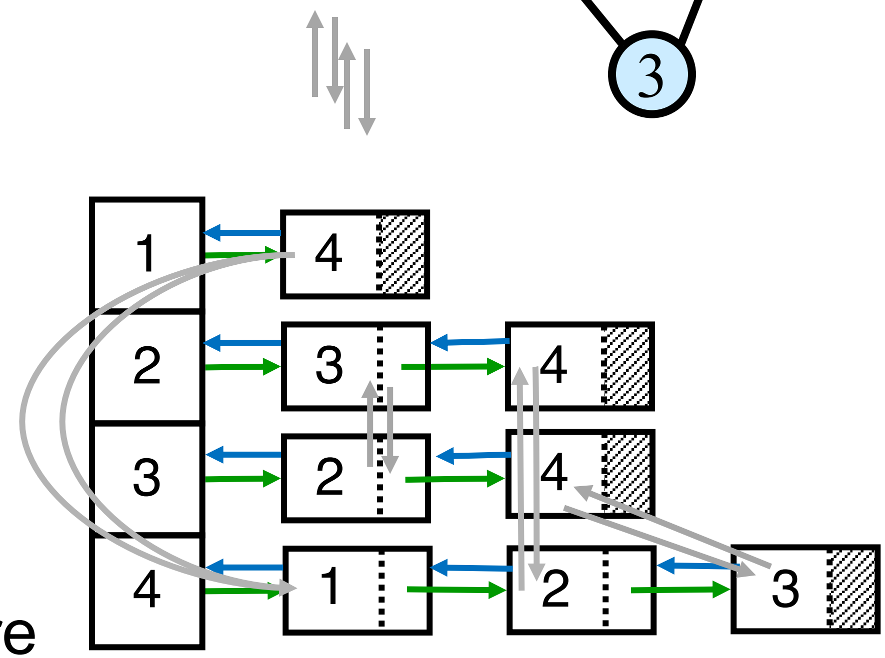
- No O(1) edge test
- More complex data structure

Storing Graphs (Internally in ALG)

Adjacency List:

O(n+m) words





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Graph Traversal

Walk (via edges) from a fixed starting vertex s to all vertices reachable from s.

- Breadth First Search (BFS): Order nodes in successive layers based on distance from s
- Depth First Search (DFS): More natural approach for exploring a maze; many efficient algs build on it.

Applications:

- Finding Connected components of a graph
- Testing Bipartiteness
- Finding Aritculation points

Breadth First Search (BFS)

Completely explore the vertices in order of their distance from s.

Three states of vertices:

- Undiscovered
- Discovered
- Fully-explored

Naturally implemented using a queue

The queue will always have the list of Discovered vertices

BFS implementation

Global initialization: mark all vertices "undiscovered"

```
BFS(s)
    mark  s "discovered"
    queue = { s }
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u fully-explored
```