Claim. In any graph $G$, $\sum v \in V \deg_G(v) = 2 |V|$. 

**Pf. Cl.** For any graph $G$ with $1k$ edges, $\sum \deg_G(v) = 2k$. 

**Base Case** $k=0$. $\sum \deg_G(v) = 0 = 2 \cdot 0 = 0$. 

**IH.** For any graph with $k-1$ edges, claim holds. 

**IS.** Suppose $G$ has edges. Remove an edge $\{x,y\}$ and call the new graph $G'$. 

By IH, $\sum_{v \in V} \deg_{G'}(v) = 2(k-1)$.

\[ \forall v, x, y: \deg_G(v) = \deg_{G'}(v). \quad 1 + \deg_G(x) = \deg_{G'}(x) \quad 1 + \deg_G(y) = \deg_{G'}(y) \]

\[ \Rightarrow \sum \deg_G(v) = 2 + \sum \deg_{G'}(v) = 2 + 2(k-1) = 2k. \]

Claim. Let $G$ be a graph with no cycles. Then $G$ has a vertex of degree $\leq 1$. 

**Pf. (by contradiction)** 

Suppose $\forall v \deg(v) > 2$. Goal: $G$ has a cycle.

\[ v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \ldots \rightarrow v_i \]

If $v_i$ is connected to $v_1$, $v_{i-2} \Rightarrow G$ has a cycle not possible. 

By assumption, $\forall v \deg(v) \geq 2 \Rightarrow v_i$ has a neighbor other than $v_{i-1}$. Call that $v_{i+1}$. 

$G$ has finite size $\Rightarrow$ this cannot continue indefinitely. 

$\Rightarrow$ either we get a cycle or a vertex of degree 1.
Claim. Show that any tree with $n$ vertices must have $n-1$ edges.

IS. $T$ with $n$ vertices. Add a new vertex and connect it to one of vertices of $T$. $T'$ has $n+1$ vertices and $n$ edges. Incorrect! Not clear if you can construct all trees this way.

IS. Start with $T$ with $n+1$ vertices. We know $T$ has a leaf, say $x$. Remove $x$ from $T$ and call the remaining graph $T'$. $T'$ has $n$ vertices.

Claim $T'$ is a tree:
- No cycle. We cannot create cycles by removing edges $T$ has no cycle $\Rightarrow$ $T'$ has no cycle
- Connected: No path in $T$ goes through $x$ (since $x$ is a leaf) $T$ is con $\Rightarrow$ $T'$ connected.

$IJT$. $T'$ is a tree $\Rightarrow$ by $IHT$ $T'$ has $n-1$ edges $\Rightarrow$ $T$ has $n$ edges.

$IHT$. Every tree with $n$ vertices has $n-1$ edges.