CSE 421

Polytime Max Flow
Linear Programming

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Summary

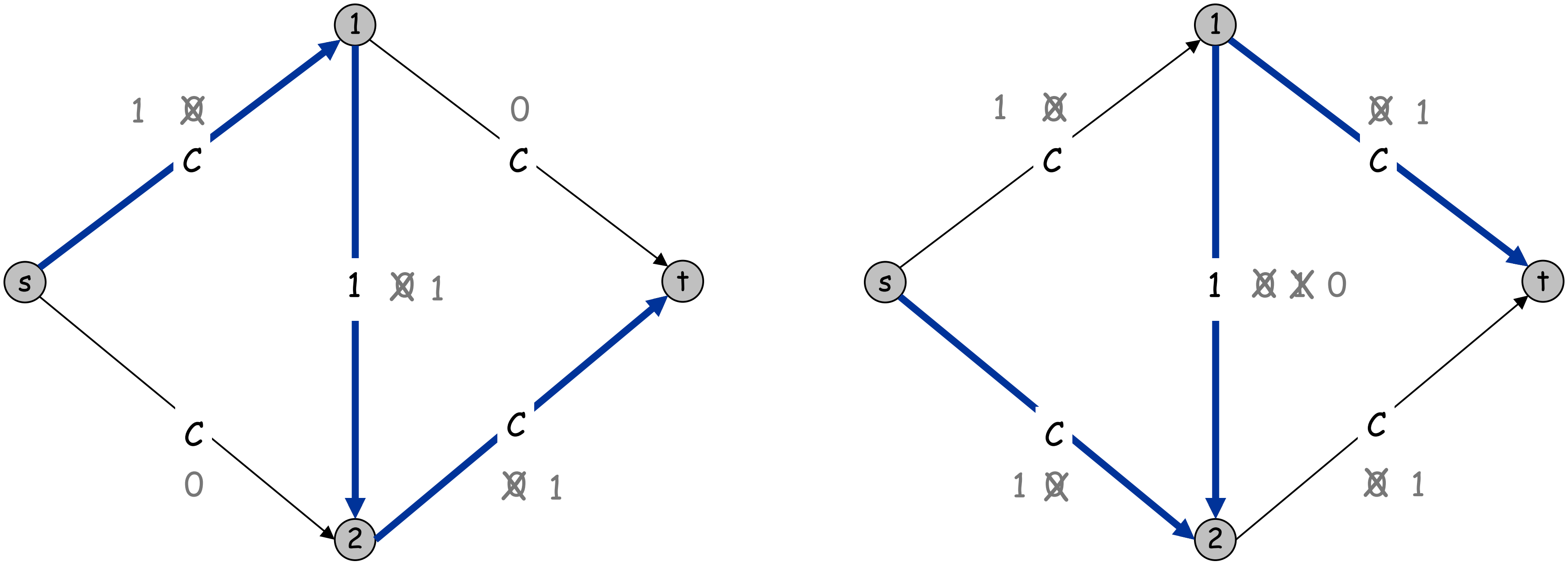
- If a problem is NP-complete it does not mean that all instances are hard, e.g., Vertex-cover has a polynomial-time algorithm in trees
- We learned the crucial idea of polynomial-time reduction. This can be even used in algorithm design, e.g., we know how to solve max-flow so we reduce image segmentation to max-flow
- NP-Complete problems are the hardest problem in NP
- NP-hard problems may not necessarily belong to NP.
- Polynomial-time reductions are transitive relations

Ford-Fulkerson: Exponential Number of Augmentations

Q. Is generic Ford-Fulkerson algorithm polynomial in input size?

m, n, and log C

A. No. If max capacity is C, then algorithm can take C iterations.



Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.

Goal: choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

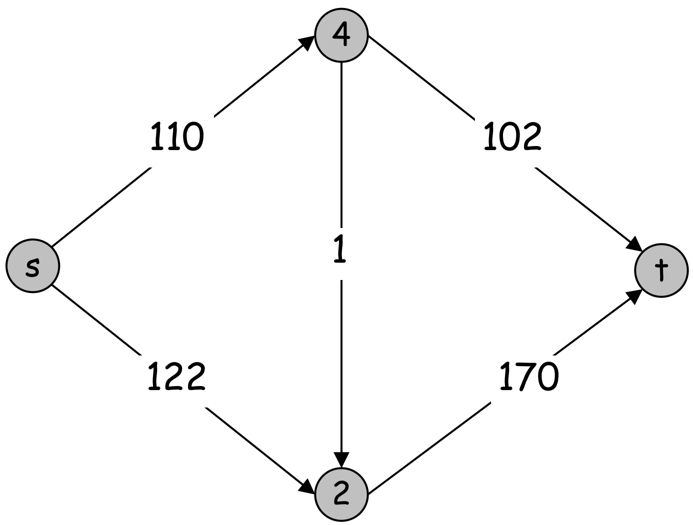
Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]

- Max bottleneck capacity.
- Sufficiently large bottleneck capacity.
- Fewest number of edges.

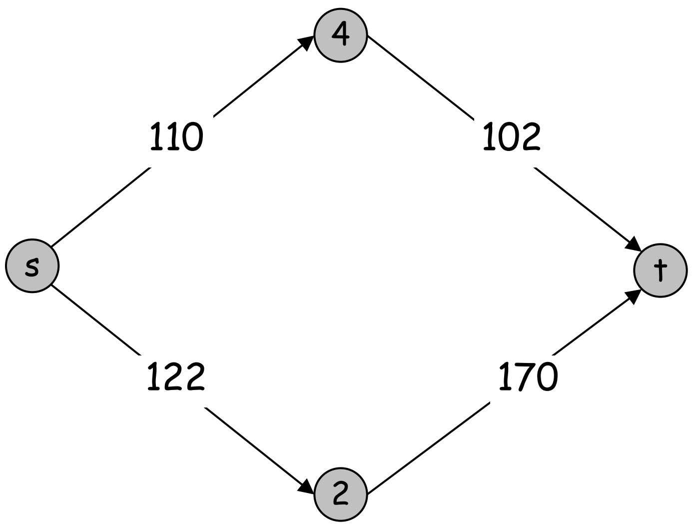
Capacity Scaling

Intuition. Choosing path with highest bottleneck capacity increases flow by max possible amount.

- Don't worry about finding exact highest bottleneck path.
- Maintain scaling parameter Δ.
- Let $G_f(\Delta)$ be the subgraph of the residual graph consisting of only arcs with capacity at least Δ.



G_f



$G_f(100)$

Capacity Scaling

```
Scaling-Max-Flow(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   Δ ← smallest power of 2 greater than or equal to C
   Gf ← residual graph

   while (Δ ≥ 1) {
      Gf(Δ) ← Δ-residual graph
      while (there exists augmenting path P in Gf(Δ)) {
         f ← augment(f, c, P)
         update Gf(Δ)
      }
      Δ ← Δ / 2
   }
   return f
}
```

Capacity Scaling: Correctness

Assumption. All edge capacities are integers between 1 and C.

Integrality invariant. All flow and residual capacity values are integral.

Correctness. If the algorithm terminates, then f is a max flow.
Pf.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths. ▪

Capacity Scaling: Running Time

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then value of the maximum flow is at most $v(f) + m\,\Delta$.

Pf. (almost identical to proof of max-flow min-cut theorem)

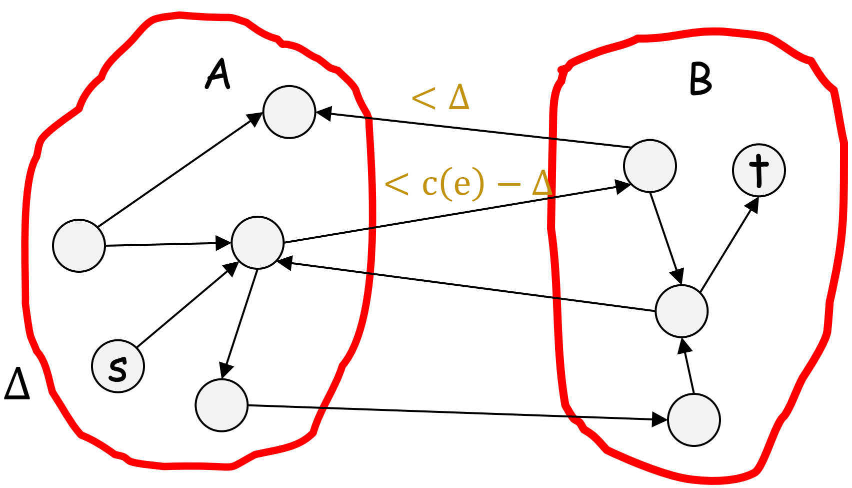
- We find a cut (A, B) such that $cap(A, B) \leq v(f) + m\,\Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of A, $s \in A$.
- By definition of f, $t \notin A$.

$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

$$\geq \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta$$

$$= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta$$

$$\geq cap(A, B) - m\Delta$$



original network

Capacity Scaling: Running Time

Lemma 1. The outer while loop repeats $1 + \lceil \log_2 C \rceil$ times.
Pf. Initially $C \leq \Delta < 2C$. Δ decreases by a factor of 2 each iteration. ▪

Lemma 2. There are at most 2m augmentations pe~~r~~ scaling phase.
- Let f be the flow at the end of the previous scaling phase.
- L2 $\Rightarrow$ $v(f^*) \leq v(f) + m\,(2\Delta)$.
- Each augmentation in a Δ-phase increases v(f) by at least Δ. ▪

Theorem. The scaling max-flow algorithm finds a max flow in $O(m \log C)$ augmentations. It can be implemented to run in $O(m^2 \log C)$ time, when m >n. ▪

Linear Programming

Linear System of Equations

In high school we learn Gaussian elimination algorithm to solve a system of linear equations

$$\begin{aligned} x_1 + x_3 &= 7 \\ 2x_2 + x_1 &= 5 \\ 3x_1 + 7x_2 - x_3 &= 1 \end{aligned}$$

We set $x_3 = 7 - x_1$ and we substitute in the following equations.

Then we substitute $x_2 = \frac{5-x_1}{2}$ in to the third equations.

The third equational uniquely defines x_1

Linear Programming

Optimize a linear function subject to linear inequalities

$$\begin{aligned}
\max \quad & 3x_1 + 4x_3 \\
s.t., \quad & x_1 + x_2 \leq 5 \\
& x_3 - x_1 = 4 \\
& x_3 - x_2 \geq -5 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

- We can have inequalities,
- We can have a linear objective functions

Applications of Linear Programming

Generalizes: Ax=b, 2-person zero-sum games, shortest path, max-flow, matching, multicommodity flow, MST, min weighted arborescence, …

Why significant?

- We can solve linear programming in polynomial time.
- Useful for approximation algorithms
- We can model many practical problems with a linear model and solve it with linear programming

Linear Programming in Practice:

- There are very fast implementations: IBM CPLEX, Gorubi in Python, CVX in Matlab, ….
- CPLEX can solve LPs with millions of variables/constraints in minutes

Example 1: Diet Problem

Suppose you want to schedule a diet for yourself. There are four category of food: veggies, meat, fruits, and dairy. Each category has its own (p)rice, (c)alory and (h)appiness per pound:

	veggies	**meat**	**fruits**	**dairy**
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

Linear Modeling: Consider a linear model: If we eat 0.5lb of meat, 0.2lb of fruits we will be $0.5\ h_m + 0.2\ h_f$ happy

- You should eat 1500 calories to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

Diet Problem by LP

- You should eat 1500 calaroies to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

	veggies	meat	fruits	dairy
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

$$\begin{aligned}
\max \quad & x_v h_v + x_m h_m + x_f h_f + x_d h_d \\
s.t. \quad & x_v p_v + x_m p_m + x_f p_f + x_d p_d \leq 20 \\
& x_v c_v + x_m c_m + x_f c_f + x_d c_d \leq 1500 \\
& x_v, x_m, x_f, x_d \geq 0
\end{aligned}$$

#pounds of veggies, meat, fruits, dairy to eat per day

How to Design an LP?

- Define the set of variables
- Put constraints on your variables,
  - should they be nonnegative?
- Write down the constraints
  - If a constraint is not linear try to approximate it with a linear constraint
- Write down the objective function
  - If it is not linear approximation with a linear function
- Decide if it is a minimize/maximization problem

Example 2: Max Flow

Define the set of variables

- For every edge e let x_e be the flow on the edge e

Put constraints on your variables

- $x_e \geq 0$ for all edge e (The flow is nonnegative)

Write down the constraints

- $x_e \leq c(e)$ for every edge e, (Capacity constraints)
- $\sum_{e \text{ out of } v} x_e = \sum_{e \text{ in to } v} x_e \quad \forall v \neq s, t$ (Conservation constraints)

Write down the objective function

- $\sum_{e \text{ out of } s} x_e$

Decide if it is a minimize/maximization problem

- max

Example 2: Max Flow

$$\begin{aligned}
\max \quad & \sum_{e \text{ out of } s} x_e \\
s.t. \quad & \sum_{e\ out\ of\ v} x_e = \sum_{e\ in\ to\ v} x_e && \forall v \neq s, t \\
& x_e \leq c(e) && \forall e \\
& x_e \geq 0 && \forall e
\end{aligned}$$

Q: Do we get exactly the same properties as Ford Fulkerson?
A: Not necessarily, the max-flow may not be integral

Example 3: Min Cost Max Flow

Suppose we can route 100 gallons of water from s to t.
But for every pipe edge e we have to pay $p(e)$ for each gallon of water that we send through e.

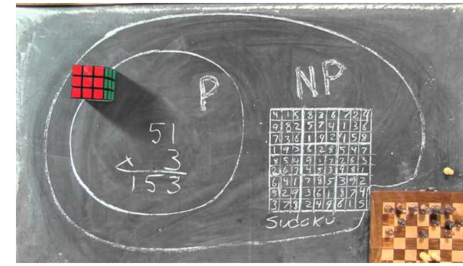
Goal: Send 100 gallons of water from s to t with minimum possible cost

$$
\begin{aligned}
\min \quad & \sum_{e \in \mathrm{E}} p(e) \cdot x_e \\
s.t. \quad & \sum_{e \text{ out of } v} x_e = \sum_{e\ in\ to\ v} x_e & \forall v \neq s, t \\
& \sum_{e \text{ out of } s} x_e = 100 \\
& x_e \leq c(e) & \forall e \\
& x_e \geq 0 & \forall e
\end{aligned}
$$

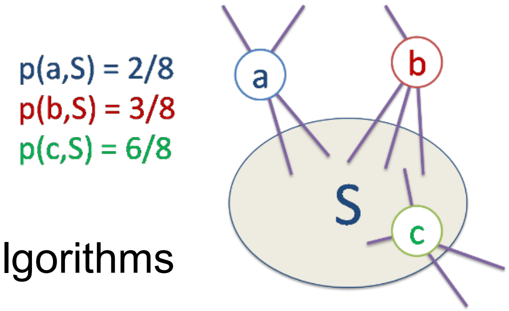
Summary (Linear Programming)

- Linear programming is one of the biggest advances in 20^{th} century
- It is being used in many areas of science: Mechanics, Physics, Operations Research, and in CS: AI, Machine Learning, Theory, …
- Almost all problems that we talked can be solved with LPs, Why not use LPs?
    - Combinatorial algorithms are typically faster
    - They exhibit a better understanding of worst case instances of a problem
    - They give certain structural properties, e.g., Integrality of Max-flow when capacities are integral
- There is rich theory of LP-duality which generalizes max-flow min-cut theorem

What is next?



- CSE 431 (Complexity Course)
  - How to prove lower bounds on algorithms?



- CSE 521 (Graduate Algorithms Course)
  - How to design streaming algorithms?
  - How to design algorithms for high dimensional data?
  - How to use matrices/eigenvalues/eigenvectors to design algorithms
  - How to use LPs to design algorithms?

- CSE 525 (Graduate Randomized Algorithms Course)
  - How to use randomization to design algorithms?
  - How to use Markov Chains to design algorithms?