Sample final is out.
Final review session on Sunday at 3:00.
Final is Monday at 2:30-4:20 at Gates 01.

CSE 421

Reductions / P vs NP

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Reductions & NP-Completeness
**Polynomial Time Reduction**

Def $A \leq_P B$: if there is an *algorithm* for problem $A$ using a ‘black box’ (subroutine) that solve problem $B$ s.t.,
- Algorithm uses only a polynomial number of steps
- Makes only a polynomial number of calls to a subroutine for $B$

So,

B is Polynomial time solvable  $\implies$  A is Polynomial time solvable

Conversely,

No efficient Algorithm for $A$  $\implies$  No efficient Algorithm for $B$

In words, $B$ is as hard as $A$ (it can be even harder)
$\leq^1_p$ Reductions

A restricted form of polynomial-time reduction often called Karp or many-to-one reduction

$A \leq^1_p B$: if and only if there is an algorithm for $A$ given a black box solving $B$ that on input $x$

- Runs for polynomial time computing an input $f(x)$ of $B$
- Makes one call to the black box for $B$ for input $f(x)$
- Returns the answer that the black box gave

We say that the function $f(.)$ is the reduction
Example 1: Indep Set $\leq_p$ Clique

**Indep Set**: Given $G=(V,E)$ and an integer $k$, is there $S \subseteq V$ s.t. $|S| \geq k$ and no two vertices in $S$ are joined by an edge?

**Clique**: Given a graph $G=(V,E)$ and an integer $k$, is there $S \subseteq V$, $|\cup| \geq k$ s.t., every pair of vertices in $S$ is joined by an edge?

**Claim**: Indep Set $\leq_p$ Clique

**Pf**: Given $G = (V, E)$ an instance of indep Set. Construct a new graph $G' = (V, E')$ where \( \{u, v\} \in E' \) if and only if \( \{u, v\} \notin E \).

\[
\begin{array}{c}
\text{S is an independ set in G} \\
\end{array}
\quad \iff \quad
\begin{array}{c}
\text{S is an Clique in G'} \\
\end{array}
\]
Example 2: Vertex Cover \( \leq_p \) Indep Set

**Vertex Cover**: Given a graph \( G=(V,E) \) and an integer \( k \), is there a vertex cover of size at most \( k \)?

**Claim**: For any graph \( G = (V, E) \), \( S \) is an independent set iff \( V - S \) is a vertex cover

**Pf**: 

\( \Rightarrow \) Let \( S \) be a independent set of \( G \). Then, \( S \) has at most one endpoint of every edge of \( G \). So, \( V - S \) has at least one endpoint of every edge of \( G \). So, \( V - S \) is a vertex cover.

\( \Leftarrow \) Suppose \( V - S \) is a vertex cover. Then, there is no edge between vertices of \( S \) (otherwise, \( V - S \) is not a vertex cover). So, \( S \) is an independent set.
Example 3: Vertex Cover $\leq_p$ Set Cover

Set Cover: Given a set $U$, collection of subsets $S_1, \ldots, S_m$ of $U$ and an integer $k$, is there a collection of $k$ sets that contain all elements of $U$?

Claim: Vertex Cover $\leq_p$ Set Cover

Pf:
Given $(G = (V, E), k)$ of vertex cover we construct a set cover input $f(G, k)$

- $U = E$
- For each $v \in V$ we create a set $S_v$ of all edges connected to $v$

This clearly is a polynomial-time reduction

So, we need to prove it gives the right answer
Example 3: Vertex Cover $\leq_p$ Set Cover

Claim: Vertex Cover $\leq_p$ Set Cover
Pf: Given $(G = (V, E), k)$ of vertex cover we construct a set cover input $f(G, k)$

- $U = E$
- For each $v \in V$ we create a set $S_v$ of all edges connected to $v$

Vertex-Cover $(G, k)$ is yes $\Rightarrow$ Set-Cover $f(G, k)$ is yes

If a set $W \subseteq V$ covers all edges, just choose $S_v$ for all $v \in W$, it covers all $U$.

Set-Cover $f(G, k)$ is yes $\Rightarrow$ Vertex-Cover $(G, k)$ is yes

If $(S_{v_1}, ..., S_{v_k})$ covers all $U$, the set $\{v_1, ..., v_k\}$ covers all edges of $G$. 
≤^m_p Reductions

Sometimes we solve problem A by reducing it to many copies of problem B

A ≤^m_p B: if and only if there is an algorithm for A given a black box solving B that on input x

- Runs for polynomial time computing inputs f_1(x), ..., f_m(x) of B where m is a polynomial on length of x.
- Makes m calls to the black box for B for each input f_i(x)
- Returns yes if one of calls to B answers yes and no otherwise.

We say that the function f(.) is the reduction