No multivariate indu

In $OPT(i,w) = \max(OPT(i-1,w), OPT(i-1, w-w_i))$

we induct on i+w.

Base Cases. Be careful specially in code 2

CSE 421

Network Flows

Shayan Oveis Gharan

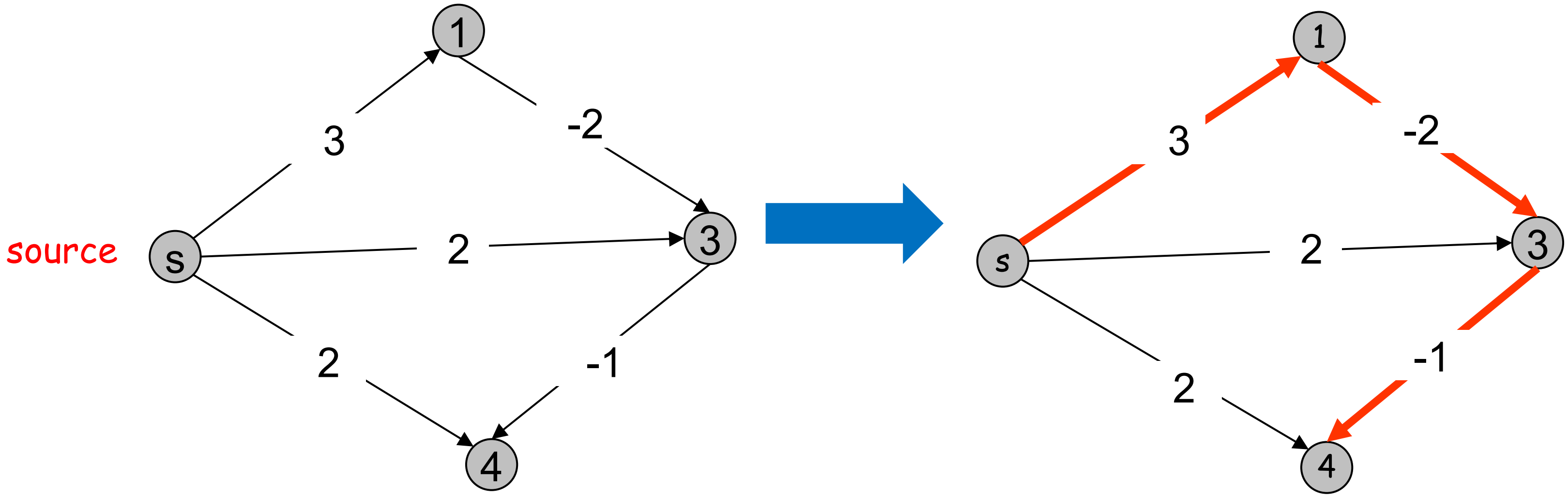
Shortest Paths with Negative Edge Weights

Shortest Paths with Neg Edge Weights

Given a weighted directed graph $G = (V, E)$ and a source vertex s, where the weight of edge (u,v) is $c_{u,v}$

Goal: Find the shortest path from s to all vertices of G.

Recall that Dikjstra's Algorithm fails when weights are negative

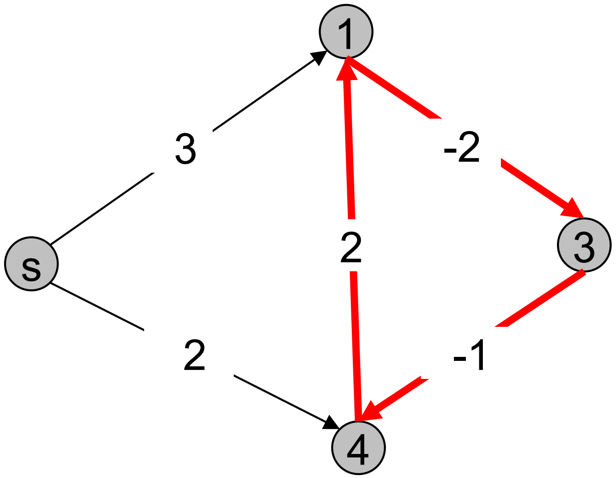


Impossibility on Graphs with Neg Cycles

Observation: No solution exists if G has a negative cycle.

This is because we can minimize the length by going over the cycle again and again.

So, suppose G does not have a negative cycle.



DP for Shortest Path

Def: Let $OPT(v,i)$ be the length of the shortest s - v path with at most i edges.

Let us characterize $OPT(v,i)$.

s ≤ i edges v

Case 1: $OPT(v,i)$ path has less than i edges.

- Then, $OPT(v,i) = OPT(v,i-1)$.

O/w last edge is from

Case 2: $OPT(v,i)$ path has exactly i edges.

- Let $s, v_1, v_2, \ldots, v_{i-1}, v$ be the $OPT(v,i)$ path with i edges.
- Then, $s, v_1, \ldots, v_{i-1}$ must be the shortest s - v_{i-1} path with at most $i-1$ edges. So,

$$OPT(v,i) = OPT(v_{i-1}, i-1) + c_{v_{i-1},v}$$

DP for Shortest Path

Def: Let $OPT(v,i)$ be the length of the shortest s - v path with at most i edges.

$$OPT(v,i) = \begin{cases} 0 & \text{if } v = s \\ \infty & \text{if } v \neq s, i = 0 \\ \min(OPT(v,i-1), \min\limits_{u:(u,v) \text{ an edge}} OPT(u,i-1) + c_{u,v}) & \end{cases}$$

So, for every v, $OPT(v, ?)$ is the shortest path from s to v.

But how long do we have to run?

Since G has no negative cycle, it has at most $n-1$ edges. So, $OPT(v, n-1)$ is the answer.

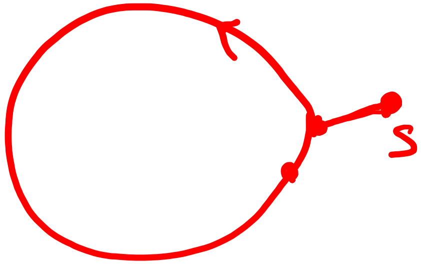
Bellman Ford Algorithm

```
for v=1 to n
   if v ≠ s then
      M[v,0]=∞
M[s,0]=0.

for i=1 to n-1
   for v=1 to n
      M[v,i]=M[v,i-1]
      for every edge (u,v)
         M[v,i]=min(M[v,i], M[u,i-1]+c_{u,v})
```

Running Time: $O(nm)$

Can we test if G has negative cycles?



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      for every edge (u,v)
         M[v,i]=min(M[v,i], M[u,i-1]+c_{u,v})
```

Running Time: $O(nm)$

Can we test if G has negative cycles?

Yes, run for i=1…3n and see if the M[v,n-1] is different from M[v,3n]

DP Techniques Summary

Recipe:

- Follow the natural induction proof.
- Find out additional assumptions/variables/subproblems that you need to do the induction
- Strengthen the hypothesis and define w.r.t. new subproblems

Dynamic programming techniques.

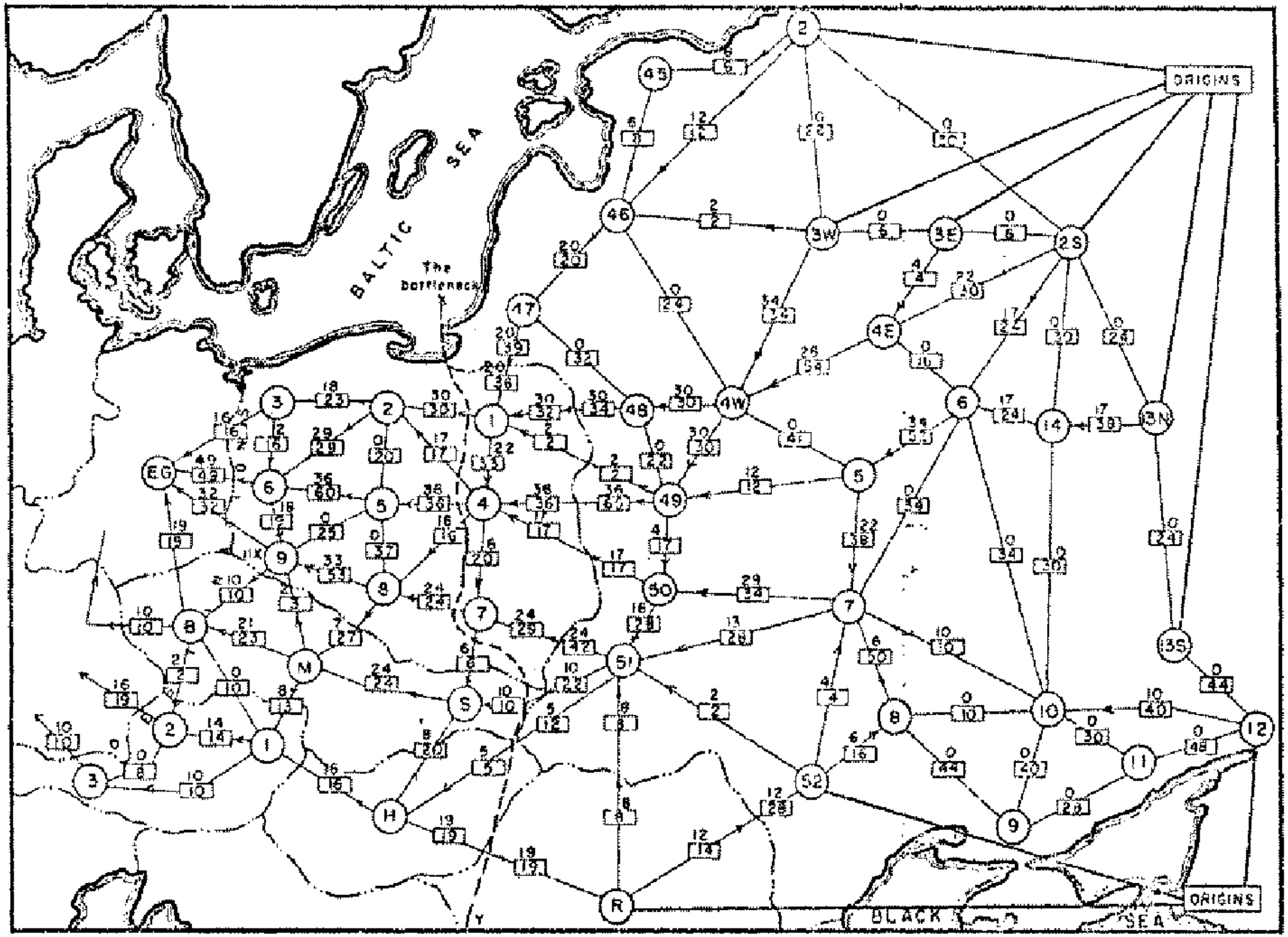
- Whenever a problem is a special case of an NP-hard problem an ordering is important:
- Adding a new variable: knapsack.
- Dynamic programming over intervals: RNA secondary structure.

Top-down vs. bottom-up:

- Different people have different intuitions
- Bottom-up is useful to optimize the memory

Network Flows

Soviet Rail Network



Reference: *On the history of the transportation and maximum flow problems*. Alexander Schrijver in Math Programming, 91: 3, 2002.

Network Flow Applications

Max flow and min cut.

- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

Nontrivial applications / reductions.

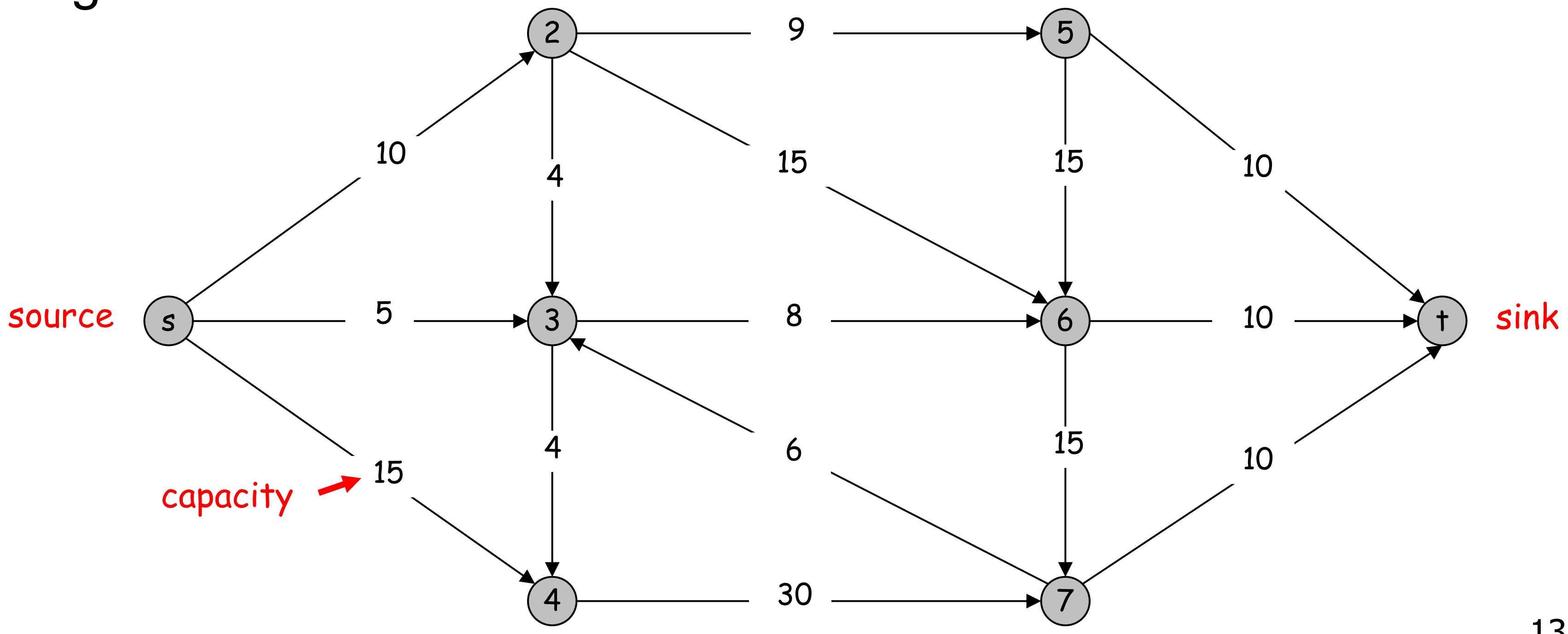
- Data mining.
- Open-pit mining.
- Project selection.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Image segmentation.
- Network connectivity.

Minimum s-t Cut Problem

Given a directed graph G = (V, E) = directed graph and two distinguished nodes: s = source, t = sink.

Suppose each directed edge e has a nonnegative capacity $c(e)$

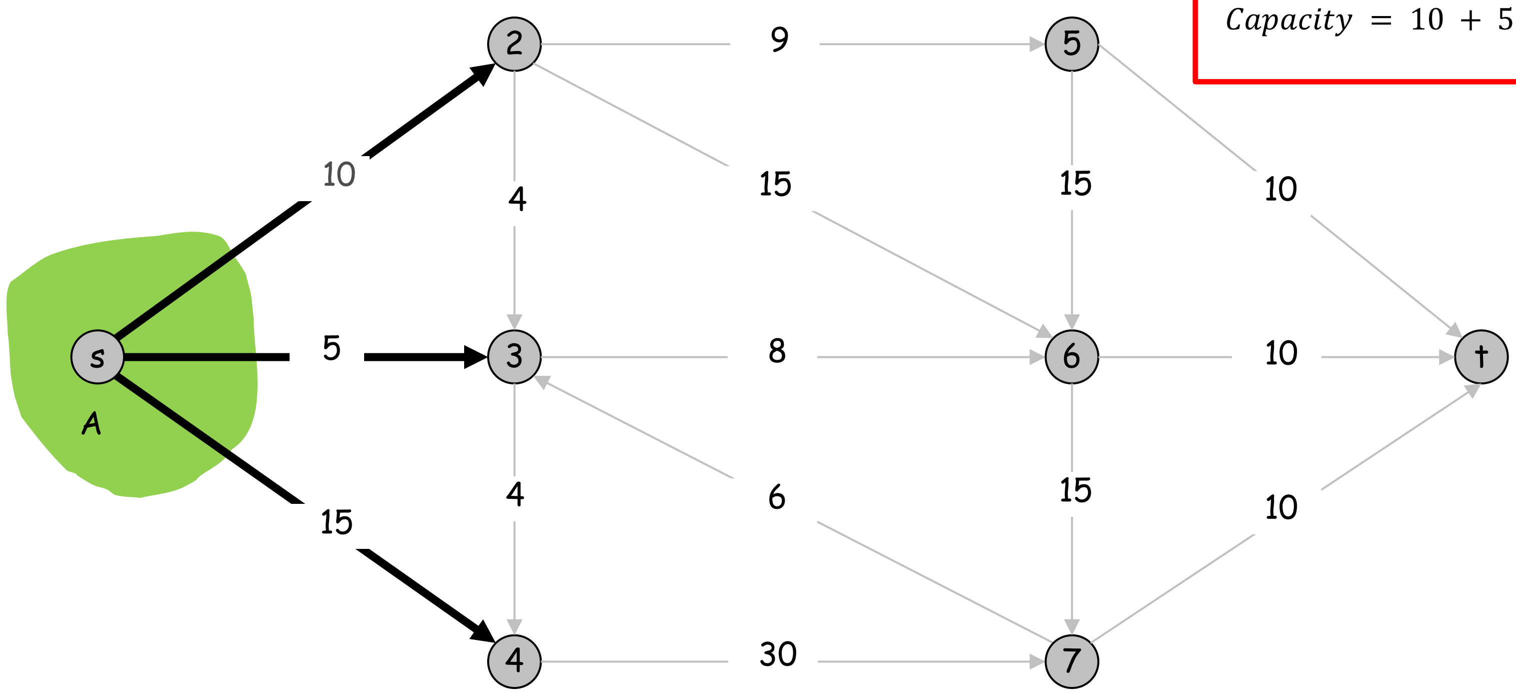
Goal: Find a cut separating s, t that cuts the minimum capacity of edges.



s-t cuts

Def. An s-t cut is a partition (A, B) of V with s ∈ A and t ∈ B.

Def. The capacity of a cut (A, B): $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



$Capacity = 10 + 5 + 15 = 30$

s-t cuts

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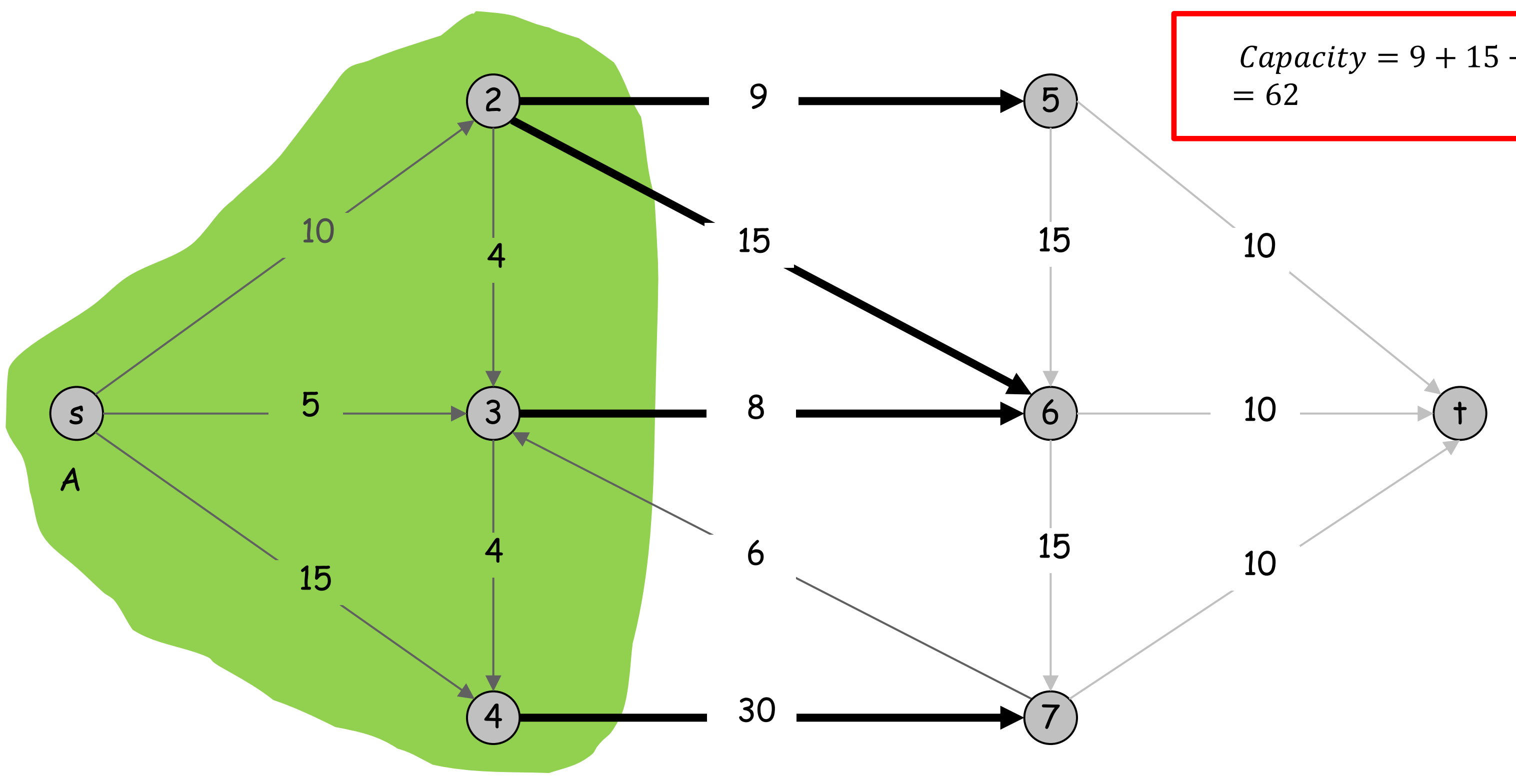


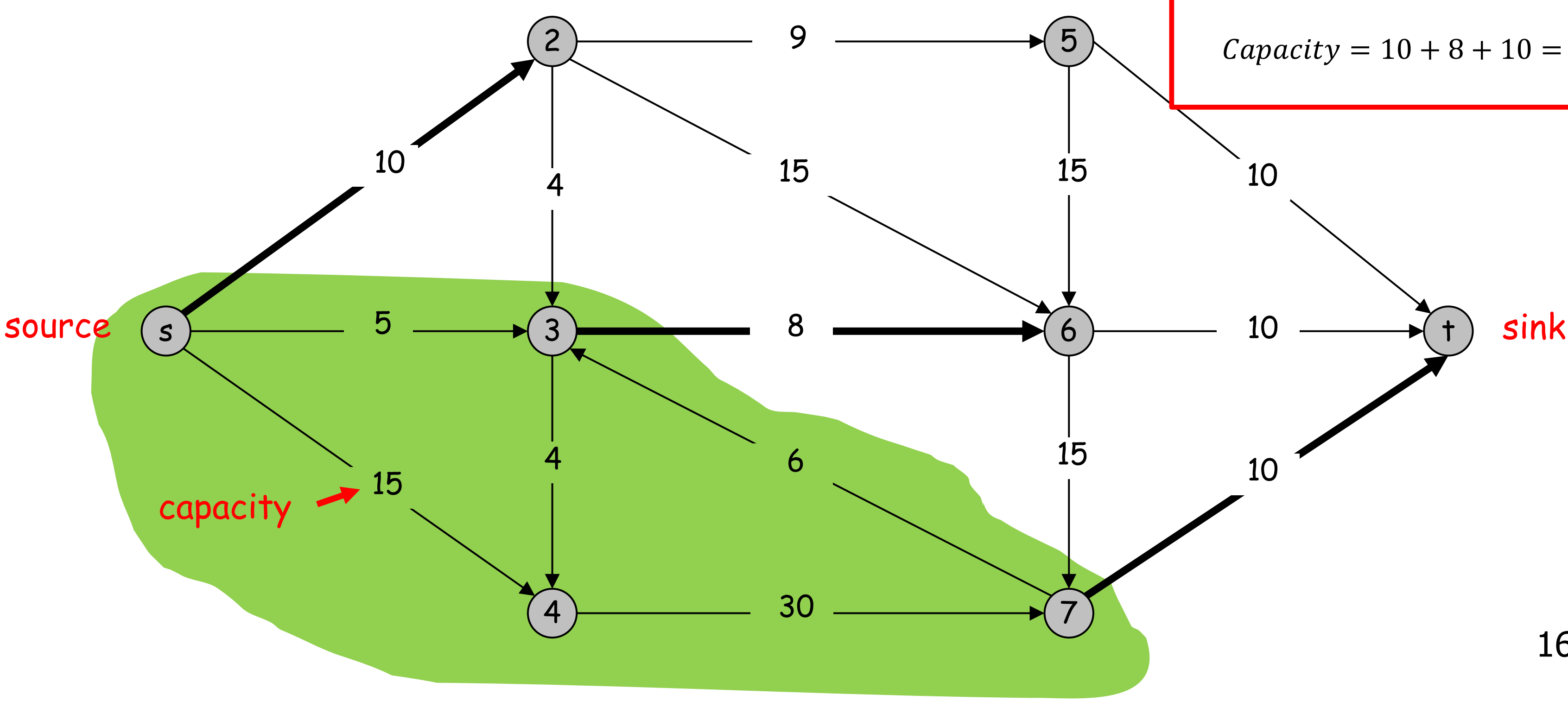
$Capacity = 9 + 15 + 8 + 30$
$= 62$

Minimum s-t Cut Problem

Given a directed graph G = (V, E) = directed graph and two distinguished nodes: s = source, t = sink.

Suppose each directed edge e has a nonnegative capacity $c(e)$

Goal: Find a s-t cut of minimum capacity



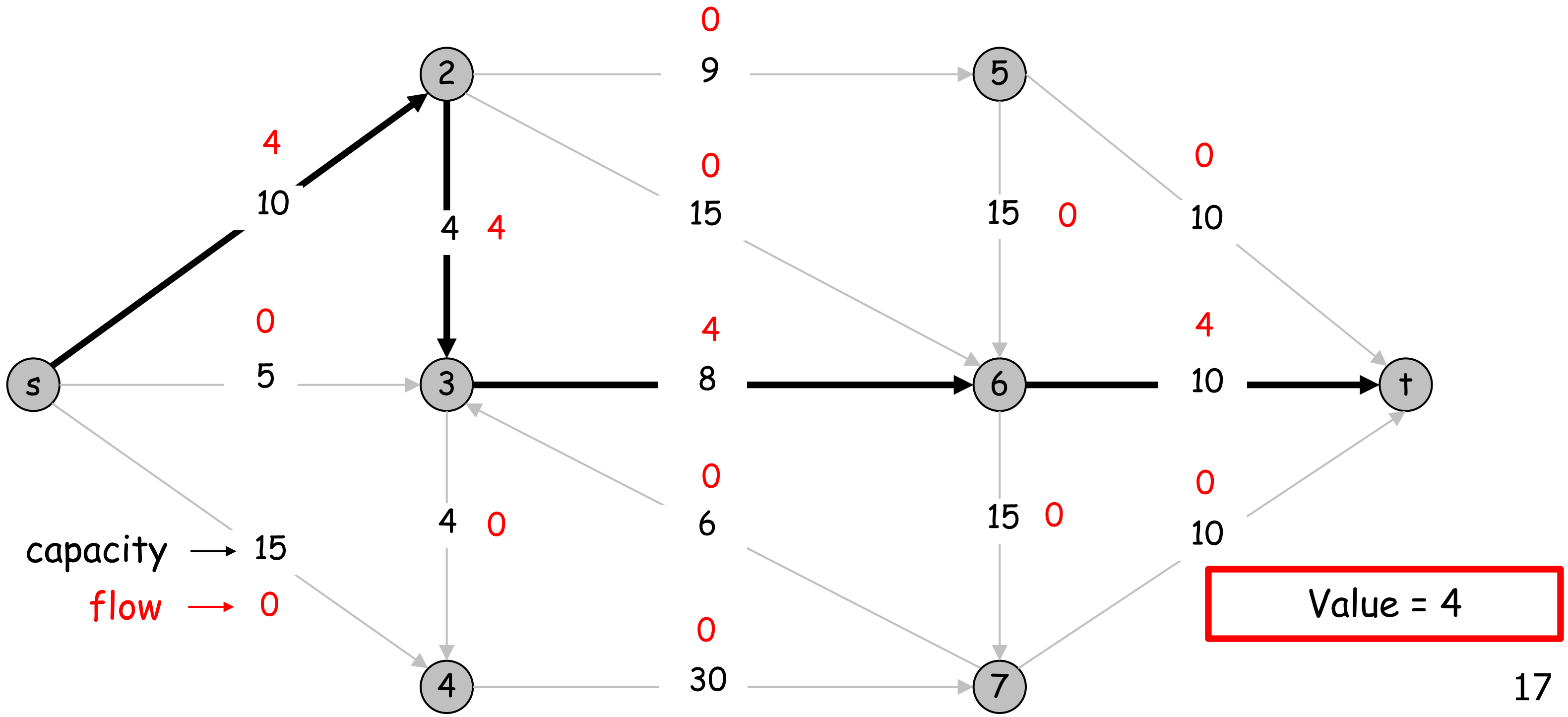
$Capacity = 10 + 8 + 10 = 28$

s-t Flows

Def. An s-t flow is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$



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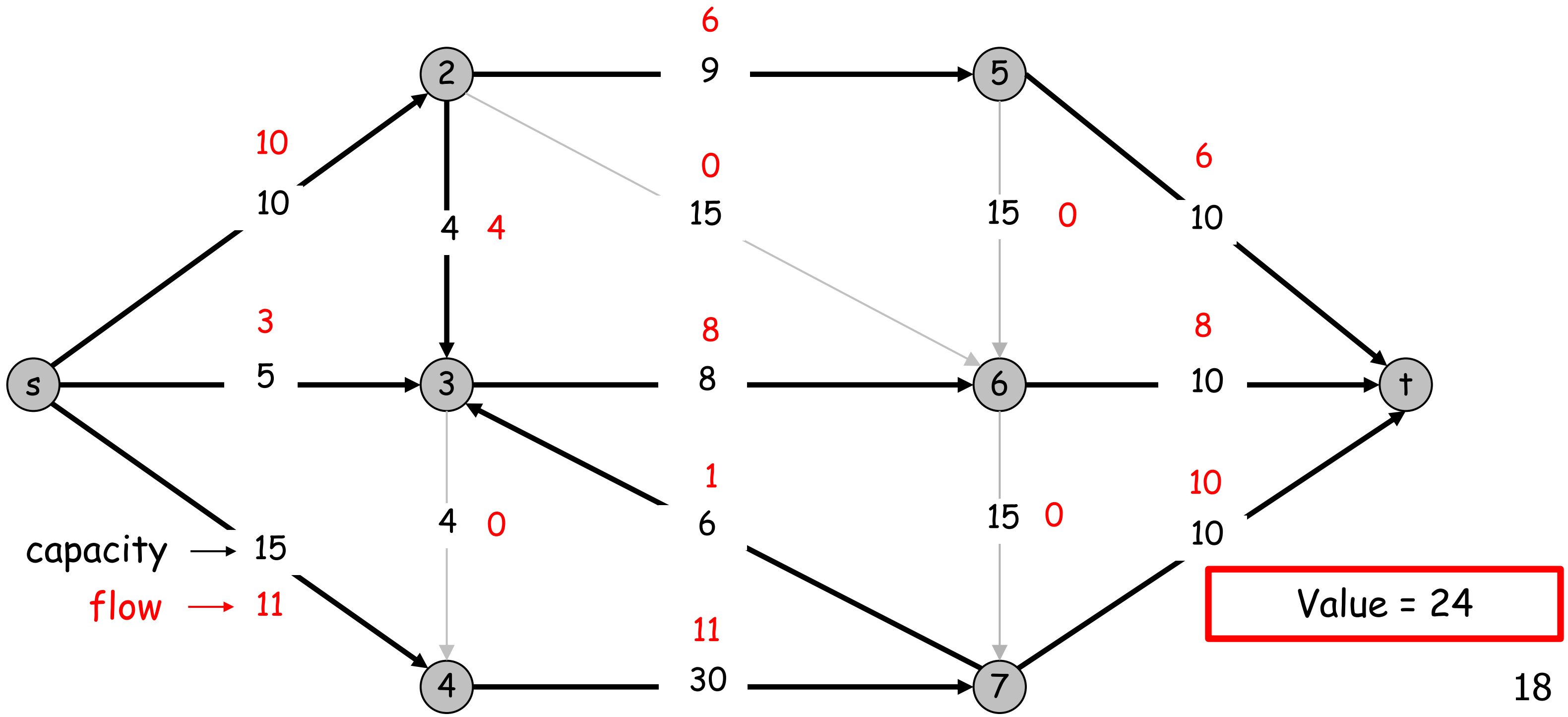
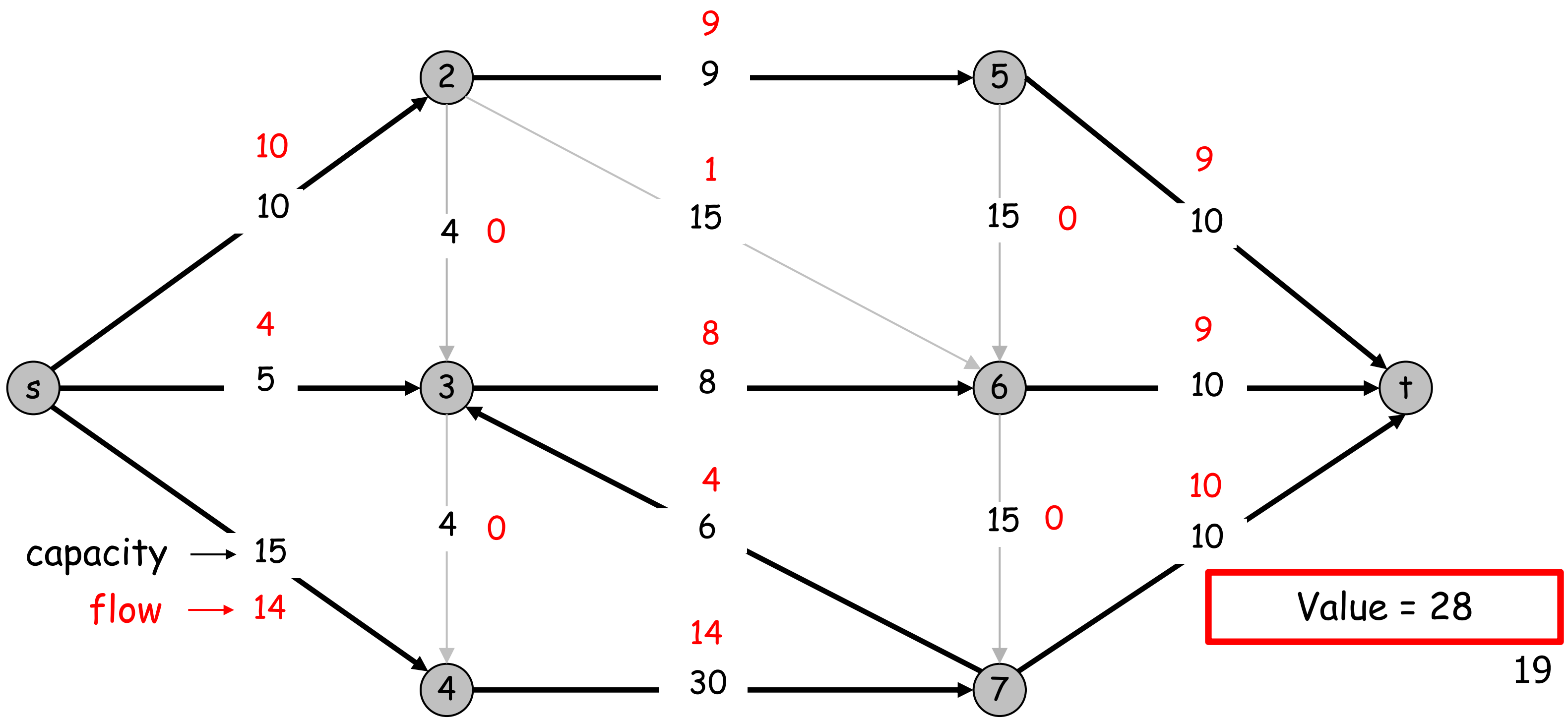


Maximum s-t Flow Problem

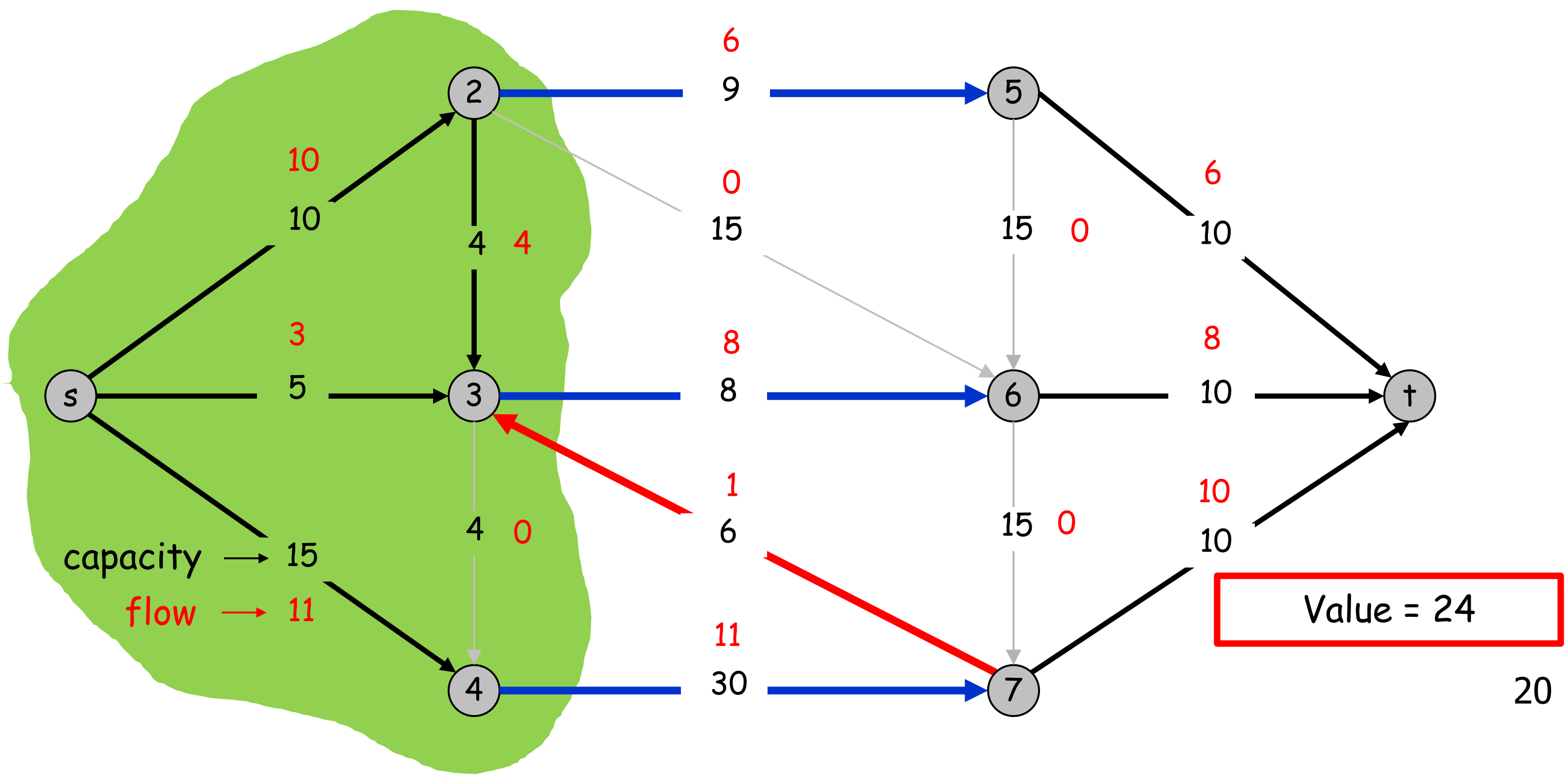
Goal: Find a s-t flow of largest value.



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



Pf of Flow value Lemma

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

By conservation of flow, all terms except v=s are0 →

$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

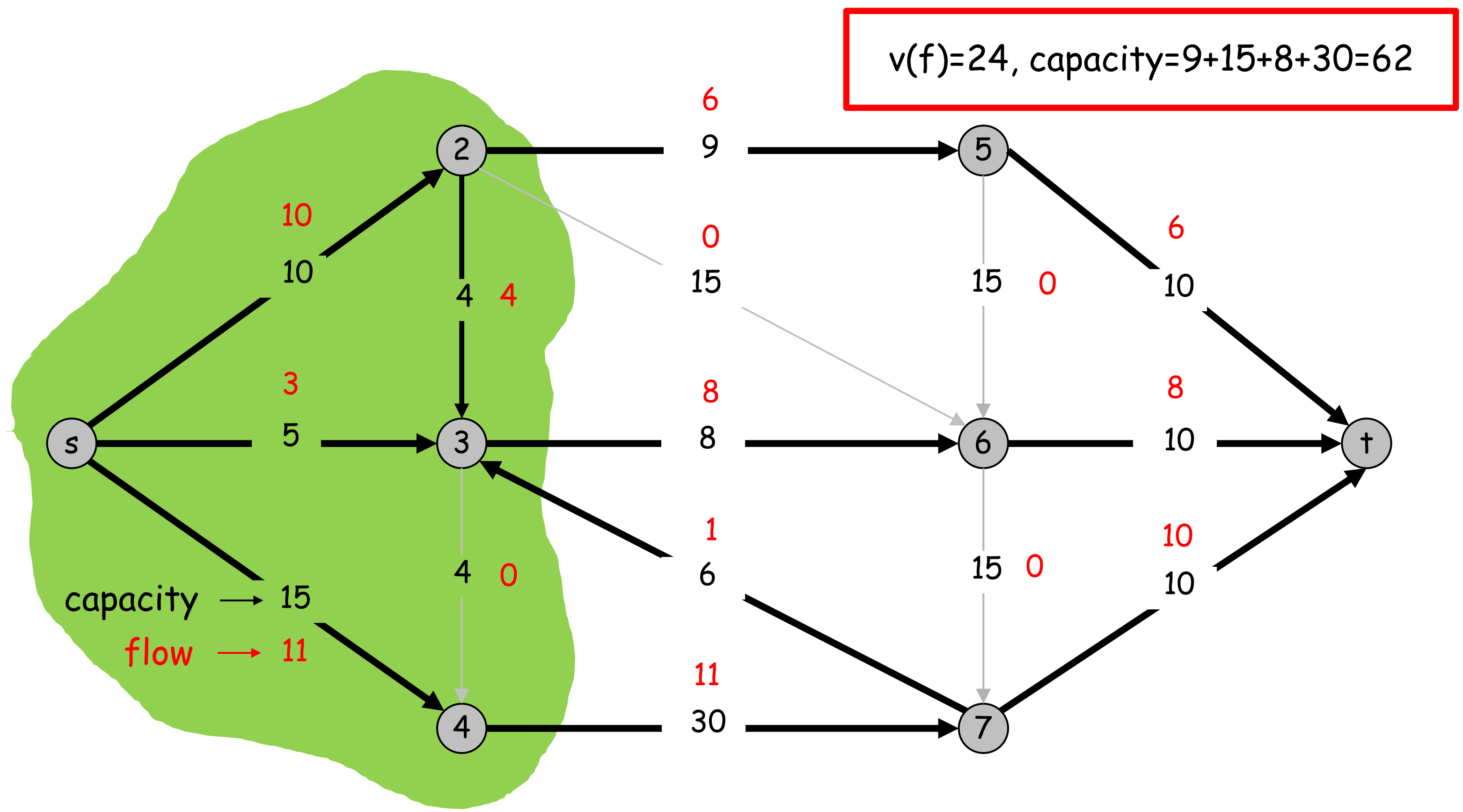
All contributions due to internal edges cancel out →

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

Weak Duality of Flows and Cuts

Cut Capacity lemma. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

$$v(f) \leq cap(A, B)$$



Weak Duality of Flows and Cuts

Cut capacity lemma. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

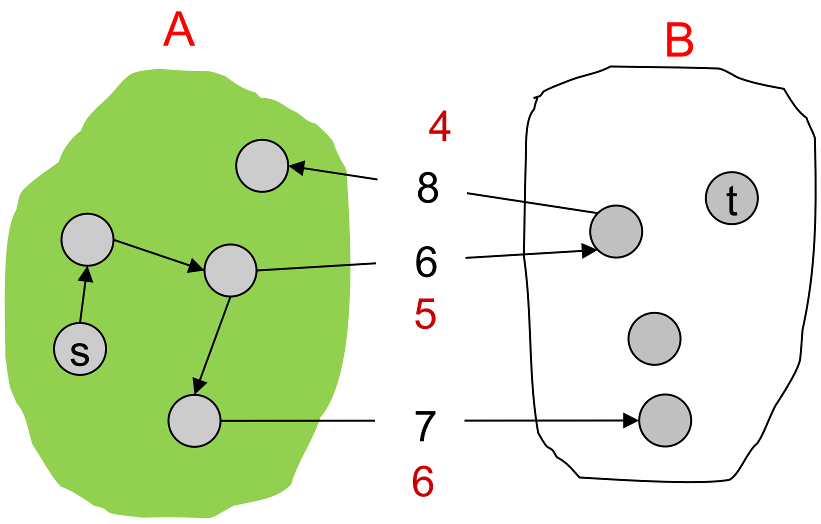
$$v(f) \leq cap(A,B)$$

Pf.

$$v(f) = \sum_{e\ out\ of\ A} f(e) - \sum_{e\ in\ to\ A} f(e)$$

$$\leq \sum_{e\ out\ of\ A} f(e)$$

$$\leq \sum_{e\ out\ of\ A} c(e) = cap(A,B)$$



Certificate of Optimality

Corollary: Suppose there is a s-t cut (A,B) such that

$$v(f) = cap(A, B)$$

Then, f is a maximum flow and (A,B) is a minimum cut.