



CSE 421

RNA Secondary Structure, Sequence Alignment, Longest Path in a DAG

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Knapsack Problem: Running Time

$$\begin{array}{l} w_1 \dots w_n \leq W \\ v_1 \dots v_n \leq V \\ W \\ \lg W \text{ bits} \end{array}$$

Running time: $\Theta(n \cdot W)$

- Not polynomial in input size!
- "Pseudo-polynomial."
- Decision version of Knapsack is NP-complete.

$$1000000 \text{ items} \quad W = 2^{1000}$$

Knapsack approximation algorithm:

There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% of optimum in time $\text{Poly}(n, \log W)$.

DP Ideas so far

- You may have to define an ordering to decrease #subproblems
- You may have to strengthen DP, equivalently the induction, i.e., you may have to carry more information to find the Optimum.
- This means that sometimes we may have to use two dimensional or three dimensional induction
- To think about DP try to characterize OPT.
- To write down your solution explicitly and carefully write $\text{OPT}(i, \dots)$

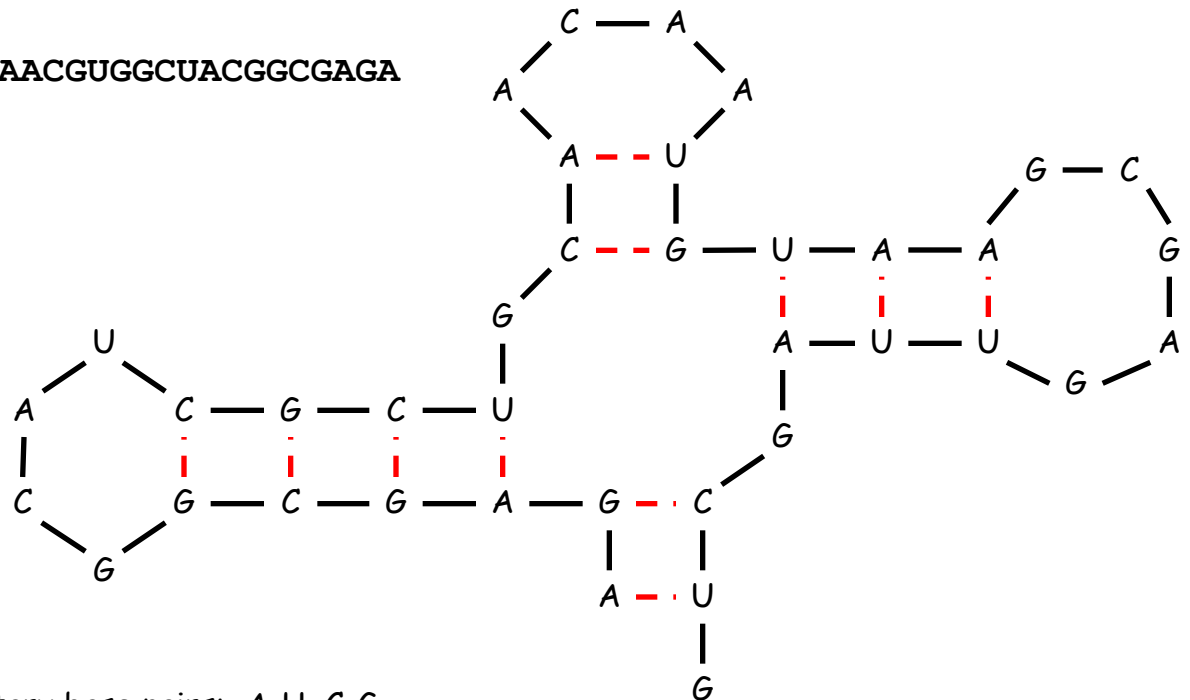
RNA Secondary Structure

RNA Secondary Structure

RNA: A String $B = b_1b_2\dots b_n$ over alphabet $\{ A, C, G, U \}$.

Secondary structure. RNA is single-stranded so it tends to loop back and form **base pairs** with itself. This structure is essential for understanding behavior of molecule.

Ex: GUCGAUUGAGCGAAUGUAACAACGUGGCUACGGCGAGA



complementary base pairs: A-U, C-G

RNA Secondary Structure (Formal)

Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

[Watson-Crick.]

- S is a *matching* and
- each pair in S is a Watson-Crick pair: A-U, U-A, C-G, or G-C.

[No sharp turns.]: The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.

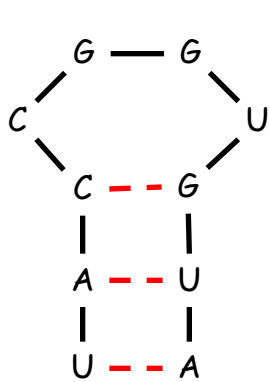
[Non-crossing.] If (b_i, b_j) and (b_k, b_l) are two pairs in S , then we cannot have $i < k < j < l$.

Free energy: Usual hypothesis is that an RNA molecule will maximize total free energy.

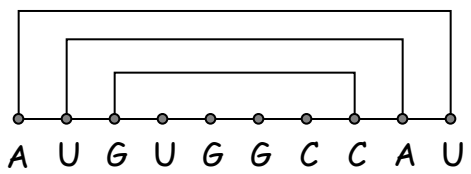
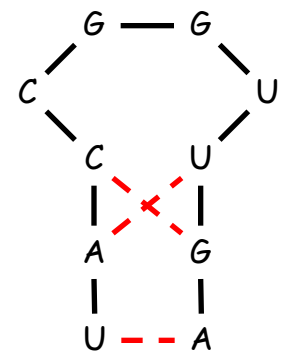
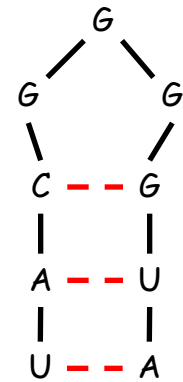
↑
approximate by number of base pairs

Goal: Given an RNA molecule $B = b_1b_2\dots b_n$, find a secondary structure S that maximizes the number of base pairs.

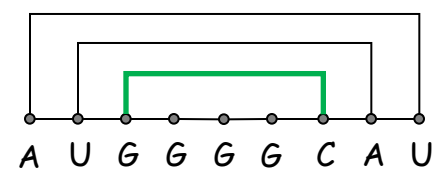
Secondary Structure (Examples)



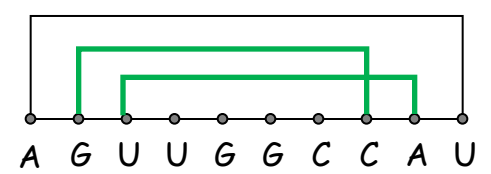
base pair



ok



~~sharp turn~~



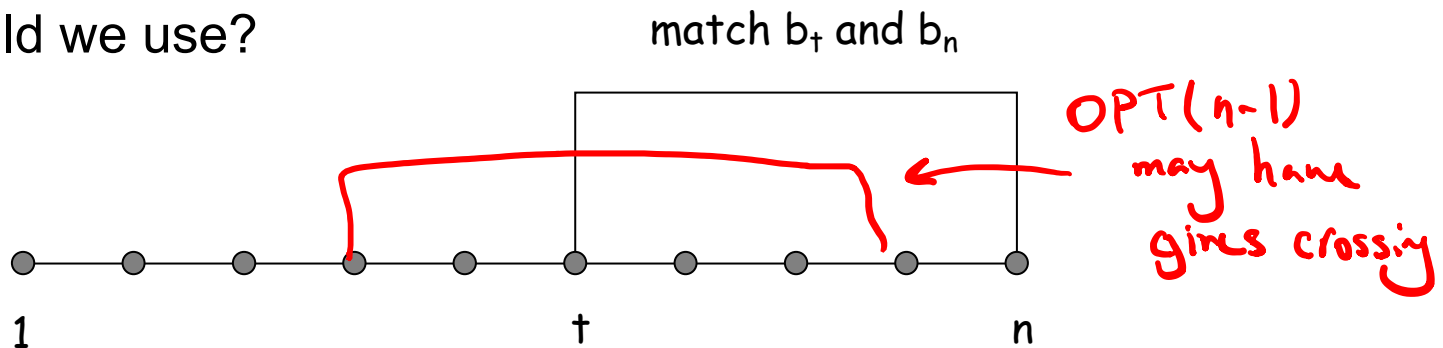
~~crossing~~

DP: First Attempt

First attempt. Let $OPT(n)$ = maximum number of base pairs in a secondary structure of the substring $b_1b_2\dots b_n$.

Suppose b_n is matched with b_t in $OPT(n)$.

What IH should we use?



Difficulty: This naturally reduces to two subproblems

- Finding secondary structure in b_1, \dots, b_{t-1} , i.e., $OPT(t-1)$
- Finding secondary structure in b_{t+1}, \dots, b_{n-1} , ???

DP: Second Attempt

Definition: $OPT(i, j)$ = maximum number of base pairs in a secondary structure of the substring b_i, b_{i+1}, \dots, b_j

The most important part of a correct DP; It fixes IH

Case 1: If $j - i \leq 4$.

- $OPT(i, j) = 0$ by no-sharp turns condition.

Case 2: Base b_j is not involved in a pair.

- $OPT(i, j) = OPT(i, j-1)$

Case 3: Base b_j pairs with b_t for some $i \leq t < j - 4$

- non-crossing constraint **decouples** resulting sub-problems
- $OPT(i, j) = \max_{t: b_j \text{ pairs with } b_t} \{ 1 + OPT(i, t - 1) + OPT(t + 1, j - 1) \}$

Recursive Code

Let $M[i,j]$ =empty for all i,j .

```
Compute-OPT(i,j){
  if (j-i <= 4)
    return 0;
  if (M[i,j] is empty)
    M[i,j]=Compute-OPT(i,j-1)
  for t=i to j-5 do
    if ( $b_t, b_j$  is in {A-U, U-A, C-G, G-C})
      M[i,j]=max(M[i,j], 1+Compute-OPT(i,t-1) +
        Compute-OPT(t+1,j-1))
  return M[j]
}
```

Does this code terminate?
What are we inducting on?

Formal Induction

Let $OPT(i, j)$ = maximum number of base pairs in a secondary structure of the substring b_i, b_{i+1}, \dots, b_j *We induct on $|i-j|$.*

Base Case: $OPT(i, j) = 0$ for all i, j where $|j - i| \leq 4$.

IH: For some $\ell \geq 4$, Suppose we have computed $OPT(i, j)$ for all i, j where $|i - j| \leq \ell$.

IS: Goal: We find $OPT(i, j)$ for all i, j where $|i - j| = \ell + 1$. Fix i, j such that $|i - j| = \ell + 1$.

Case 1: Base b_j is not involved in a pair.

- $OPT(i, j) = OPT(i, j-1)$ *[this we know by IH since $|i - (j - 1)| = \ell$]*

Case 2: Base b_j pairs with b_t for some $i \leq t < j - 4$

- $OPT(i, j) = \max_{t: b_i \text{ pairs with } b_t} \{ 1 + OPT(i, t - 1) + OPT(t + 1, j - 1) \}$

We know by IH since difference $\leq \ell$

Bottom-up DP

```
for k = 1, 2, ..., n-1
  for i = 1, 2, ..., n-1
    j = i + k
    if (j-i <= 4)
      M[i,j]=0;
    else
      M[i,j]=M[i,j-1]
      for t=i to j-5 do
        if ( $b_t, b_j$  is in {A-U, U-A, C-G, G-C})
          M[i,j]=max(M[i,j], 1+ M[i,t-1] + M[t+1,j-1])

return M[1, n]
}
```

4	0	0	0	↗
3	0	0		↗
2	0			↗
1				↗
	6	7	8	9

j

Running Time: $O(n^3)$

Lesson

We may not always induct on i or w to get to smaller subproblems.

We may have to induct on $|i - j|$ or $i + j$ when we are dealing with more complex problems, e.g., intervals

Sequence Alignment

Word Alignment

How similar are two strings?

ocurrance

occurrence

o	c	u	r	r	a	n	c	e	-
---	---	---	---	---	---	---	---	---	---

o	c	c	u	r	r	e	n	c	e
---	---	---	---	---	---	---	---	---	---

5 mismatches, 1 gap

o	c	-	u	r	r	a	n	c	e
---	---	---	---	---	---	---	---	---	---

o	c	c	u	r	r	e	n	c	e
---	---	---	---	---	---	---	---	---	---

1 mismatch, 1 gap

o	c	-	u	r	r	-	a	n	c	e
---	---	---	---	---	---	---	---	---	---	---

o	c	c	u	r	r	e	-	n	c	e
---	---	---	---	---	---	---	---	---	---	---

0 mismatches, 3 gaps

Edit Distance

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]

Cost = # of gaps + #mismatches.

Applications.

- Basis for Unix diff and Word correct in editors.
- Speech recognition.
- Computational biology.

C T G A C C T A C C T

C C T G A C T A C A T

Cost: 5

- C T G A C C T A C C T

C C T G A C - T A C A T

Cost: 3

DP for Sequence Alignment

Let $OPT(i, j)$ be min cost of aligning x_1, \dots, x_i and y_1, \dots, y_j

Case 1: OPT matches x_i, y_j

- Then, pay mis-match cost if $x_i \neq y_j$ + min cost of aligning x_1, \dots, x_{i-1} and y_1, \dots, y_{j-1} i.e., $OPT(i-1, j-1)$

Case 2: OPT leaves x_i unmatched

- Then, pay gap cost for x_i + $OPT(i-1, j)$

Case 3: OPT leaves y_j unmatched

- Then, pay gap cost for y_j + $OPT(i, j-1)$

Bottom-up DP

```
Sequence-Alignment(m, n, x1x2...xm, y1y2...yn) {  
  for i = 0 to m  
    M[0, i] = i  
  for j = 0 to n  
    M[j, 0] = j  
  
  for i = 1 to m  
    for j = 1 to n  
      M[i, j] = min( (xi=yj ? 0:1) + M[i-1, j-1],  
                    1 + M[i-1, j],  
                    1 + M[i, j-1])  
  
  return M[m, n]  
}
```

Analysis: $\Theta(mn)$ time and space.

English words or sentences: $m, n \leq 10, \dots, 20$.


Computational biology: $m = n = 100,000$. 10 billions ops OK,
but 40GB array?

Optimizing Memory

If we are not using strong induction in the DP, we just need to use the last (row) of computed values.

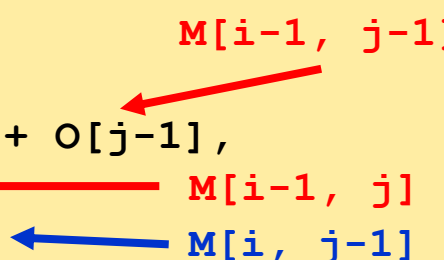
```
Sequence-Alignment(m, n, x1x2...xm, y1y2...yn) {  
  for i = 0 to m  
    M[0, i] = i  
  for j = 0 to n  
    M[j, 0] = j  
  
  for i = 1 to m  
    for j = 1 to n  
      M[i, j] = min( (xi=yj ? 0:1) + M[i-1, j-1],  
                    1 + M[i-1, j],  
                    1 + M[i, j-1])  
  
  return M[m, n]  
}
```

Just need $i - 1, i$ rows
to compute $M[i, j]$



DP with $O(m + n)$ memory

- Keep an Old array containing values of the last row
- Fill out the new values in a New array
- Copy new to old at the end of the loop

```
Sequence-Alignment(m, n, x1x2...xm, y1y2...yn) {  
  for i = 0 to m  
    O[i] = i  
  for i = 1 to m  
    N[0]=i  
    for j = 1 to n  
      N[j] = min( (xi=yj ? 0:1) + O[j-1],  
                 1 + O[j],  
                 1 + N[j-1])  
        
    for j = 1 to n  
      O[j]=N[j]  
  return N[n]  
}
```

Lesson

Advantage of a bottom-up DP:

It is much easier to optimize the space.

Longest Path in a DAG

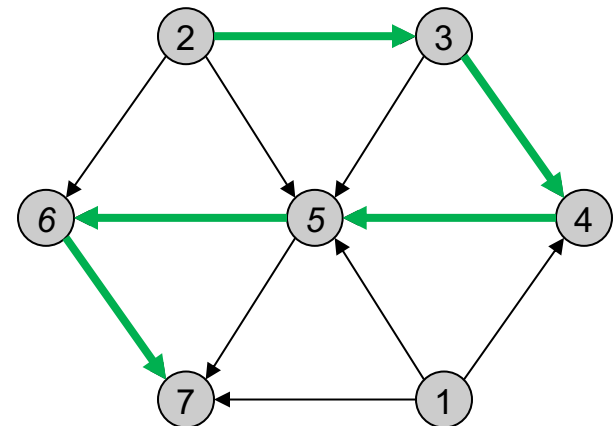
Longest Path in a DAG

Goal: Given a DAG G , find the longest path.

Recall: A directed graph G is a DAG if it has no cycle.

This problem is NP-hard for general directed graphs:

- It has the Hamiltonian Path as a special case

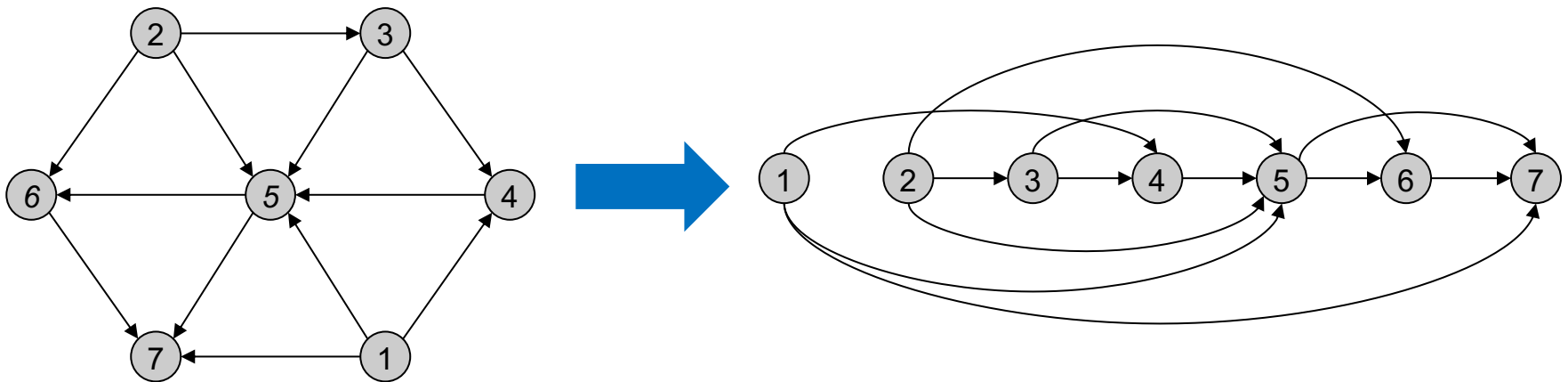


DP for Longest Path in a DAG

Q: What is the right **ordering**?

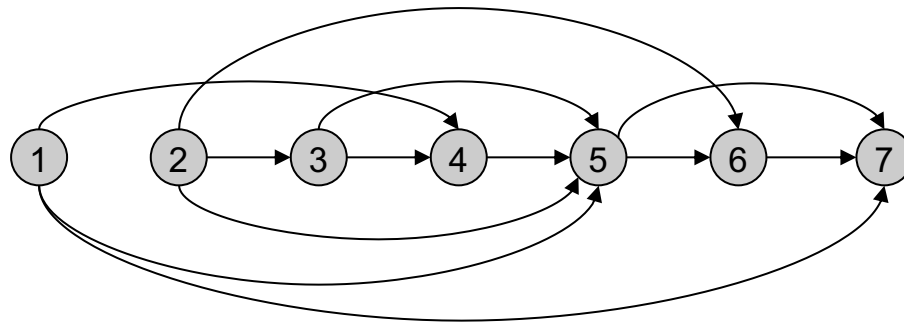
Remember, we have to use that G is a DAG, ideally in defining the ordering

We saw that every DAG has a **topological sorting**
So, let's use that as an ordering.



DP for Longest Path in a DAG

Suppose we have labelled the vertices such that (i, j) is a directed edge only if $i < j$.



Let $OPT(j)$ = length of the longest path ending at j

Suppose in the longest path ending at j , last edge is (i, j) .

Then, **none** of the $i + 1, \dots, j - 1$ are in this path since topological ordering. So,

$$OPT(j) = OPT(i) + 1.$$

DP for Longest Path in a DAG

Suppose we have labelled the vertices such that (i, j) is a directed edge only if $i < j$.

Let $OPT(j)$ = length of the longest path ending at j

$$OPT(j) = \begin{cases} 0 & \text{If } j \text{ is a source} \\ 1 + \max_{i:(i,j) \text{ an edge}} OPT(i) & \text{o.w.} \end{cases}$$

DP for Longest Path in a DAG

Let G be a DAG given with a topological sorting: For all edges (i, j) we have $i < j$.

```
Compute-OPT(j) {
  if (in-degree(j) == 0)
    return 0
  if (M[j] == empty)
    M[j] = 0;
  for all edges (i, j)
    M[j] = max(M[j], 1 + Compute-OPT(i))
  return M[j]
}
Output max(M[1], ..., M[n])
```

Running Time: $O(n + m)$

Memory: $O(n)$

Can we output the longest path?

Outputting the Longest Path

Let G be a DAG given with a topological sorting: For all edges (i, j) we have $i < j$.

Initialize $\text{Parent}[j] = -1$ for all j .

Compute-OPT(j) {

if ($\text{in-degree}(j) == 0$)

return 0

if ($M[j] == \text{empty}$)

$M[j] = 0$;

for all edges (i, j)

if ($M[j] < 1 + \text{Compute-OPT}(i)$)

$M[j] = 1 + \text{Compute-OPT}(i)$

Parent[j]= i

return $M[j]$

}

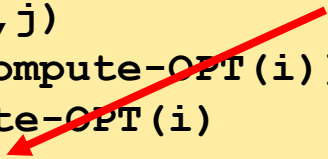
Let $M[k]$ be the maximum of $M[1], \dots, M[n]$

While ($\text{Parent}[k] \neq -1$)

 Print k

$k = \text{Parent}[k]$

Record the entry that
we used to compute $\text{OPT}(j)$



DP Techniques Summary

Recipe:

- Follow the natural induction proof.
- Find out additional assumptions/variables/subproblems that you need to do the induction
- Strengthen the hypothesis and define w.r.t. new subproblems

Dynamic programming techniques.

- Whenever a problem is a special case of an NP-hard problem an ordering is important:
- Adding a new variable: knapsack.
- Dynamic programming over intervals: RNA secondary structure.

Top-down vs. bottom-up:

- Different people have different intuitions
- Bottom-up is useful to optimize the memory