IH: Assume $x_i \cdots x_j$ is max of $x_1 \cdots x_{n-1}$ (inducting on steps of ALG)

$x_k \cdots x_{n-1}$ is max suff of $x_1 \cdots x_{n-1}$.

ALG: If $x_n + (x_k \cdots x_{n-1}) > x_1 + \cdots x_j$ then $x_k \cdots x_n$ is new OPT.

IS. Supp OPT of $x_1 \cdots x_n$ is $x_a \cdots x_b$.

Case 1: $b < n$. Then $x_a \cdots x_b$ is also OPT of $x_1 \cdots x_{n-1}$.

By IH, we must have $a = i$, $b = j$. done ✓.

[There could be multiple OPT but assume one for simplicity]

Case 2: $b = n$. ©

Claim: $x_a \cdots x_{b-1}$ is OPT suffix of $x_1 \cdots x_{n-1}$.

Pf. If not $x_k + \cdots x_{n-1} + x_n > x_a + \cdots x_{b-1} + x_n = x_b$ is not OPT contradiction □

$\Rightarrow x_a \cdots x_{b-1}$ is OPT suffix so $a = k$ (by IH). So we update OPT correctly.

ALG: If $x_k + \cdots + x_n \leq 0$ then OPT suff is empty, otherwise add x_n to $x_k \cdots x_{n-1}$.

Supp $x_a \cdots x_n$ is OPT suffix (it can be empty).

Case 1: If OPT-suff is empty.

In this case we must have $x_k + \cdots x_n \leq 0$ (otherwise OPT suffix is positive). And we update correctly.

Case 2: OPT-suffix is non-empty.

Claim: $x_a \cdots x_{n-1}$ is OPT-suff of $x_1 \cdots x_{n-1}$.

Pf. If not, $x_k + \cdots + x_{n-1} + x_n > x_a + \cdots + x_{n-1} + x_n$ is not OPT-suff contradiction. □

Therefore by IH, we must have $k = a$.

And $x_{k}+\cdots x_n = x_a+\cdots x_n > 0$ So we update correctly.