CSE 421

Set Cover, Alg Design by Induction

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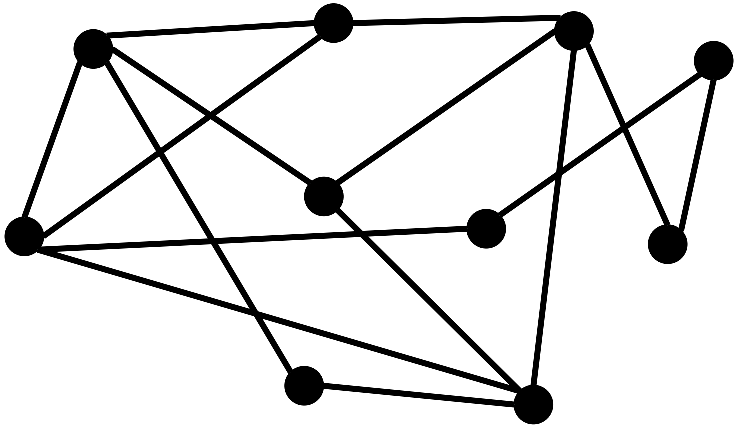
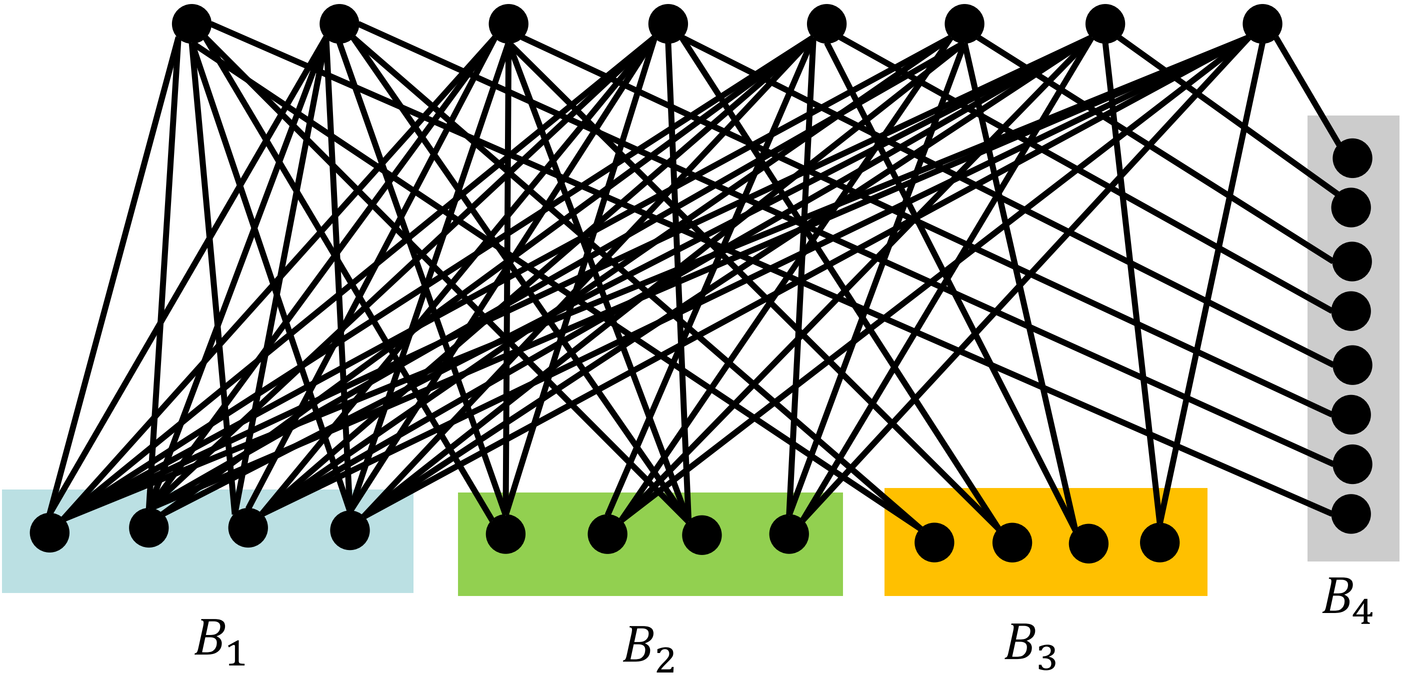
A Different Greedy Rule

Greedy 2: Iteratively, pick both endpoints of an uncovered edge.

Vertex cover = 6



Greedy 2: Pick Both endpoints of an uncovered edge



Greedy vertex cover = 16

OPT vertex cover = 8

Greedy (2) gives 2-approximation

Thm: Size of greedy (2) vertex cover is at most twice as big as size of optimal cover

Pf: Suppose Greedy (2) picks endpoints of edges $e_1, \ldots, e_k$. Since these edges do not touch, every valid cover must pick one vertex from each of these edges!

i.e., $OPT \geq k$.

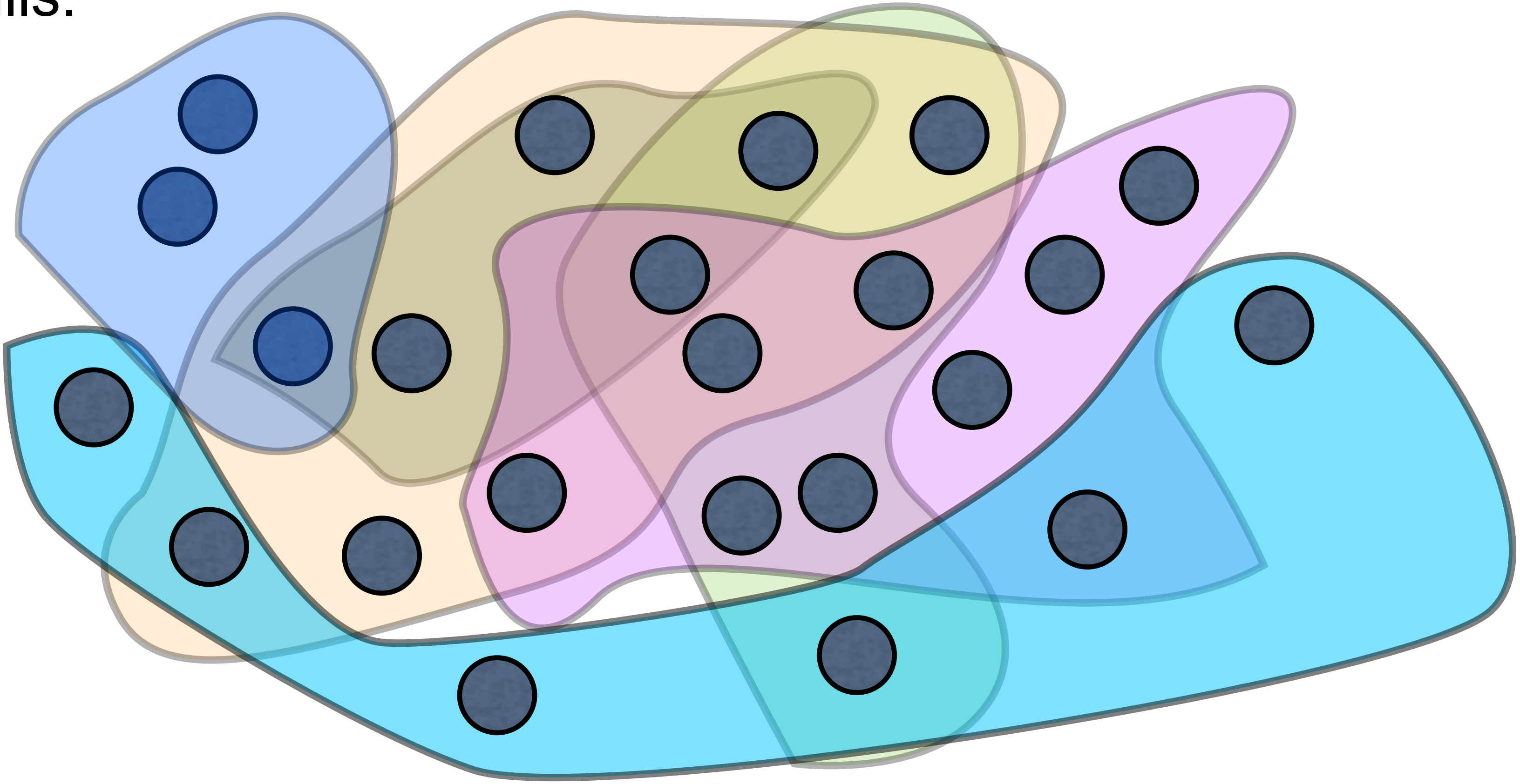
But the size of greedy cover is 2k. So, Greedy is a 2-approximation.

Set Cover

Given a number of sets on a ground set of n elements,

Goal: choose minimum number of sets that cover all.

e.g., a company wants to hire employees with certain skills.

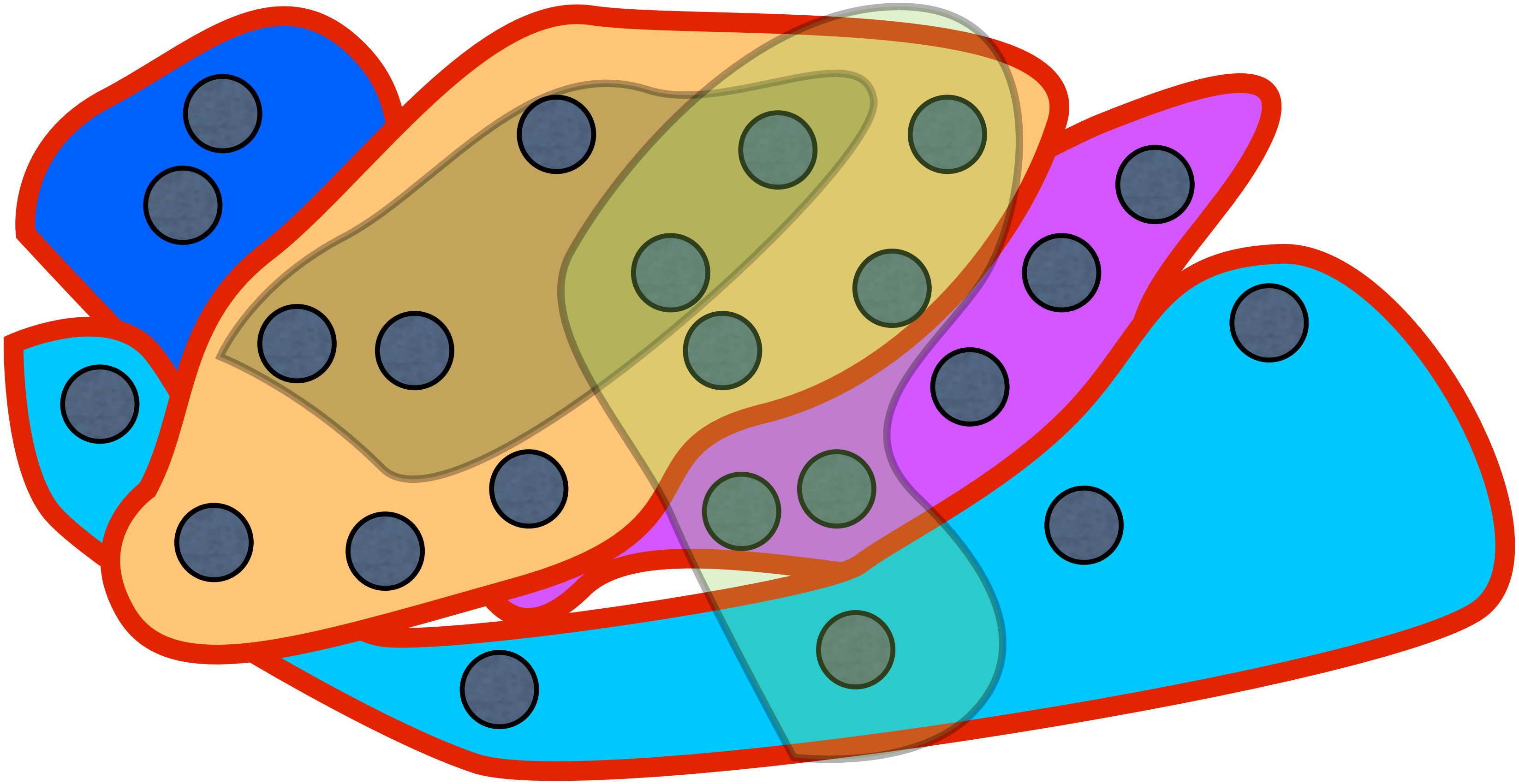


Set Cover

Given a number of sets on a ground set of elements,

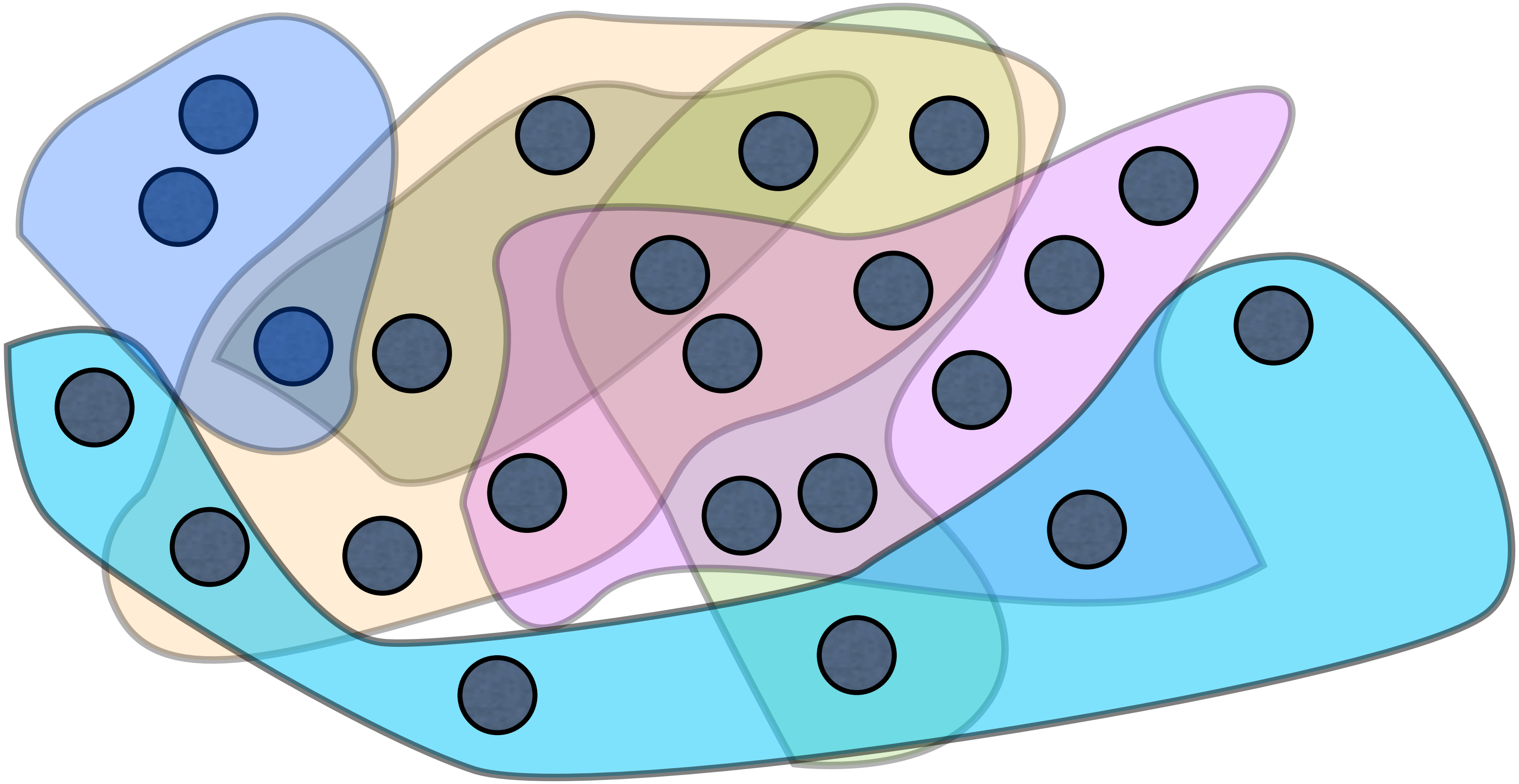
Goal: choose minimum number of sets that cover all.

Set cover = 4



A Greedy Algorithm

Strategy: Pick the set that maximizes # new elements covered



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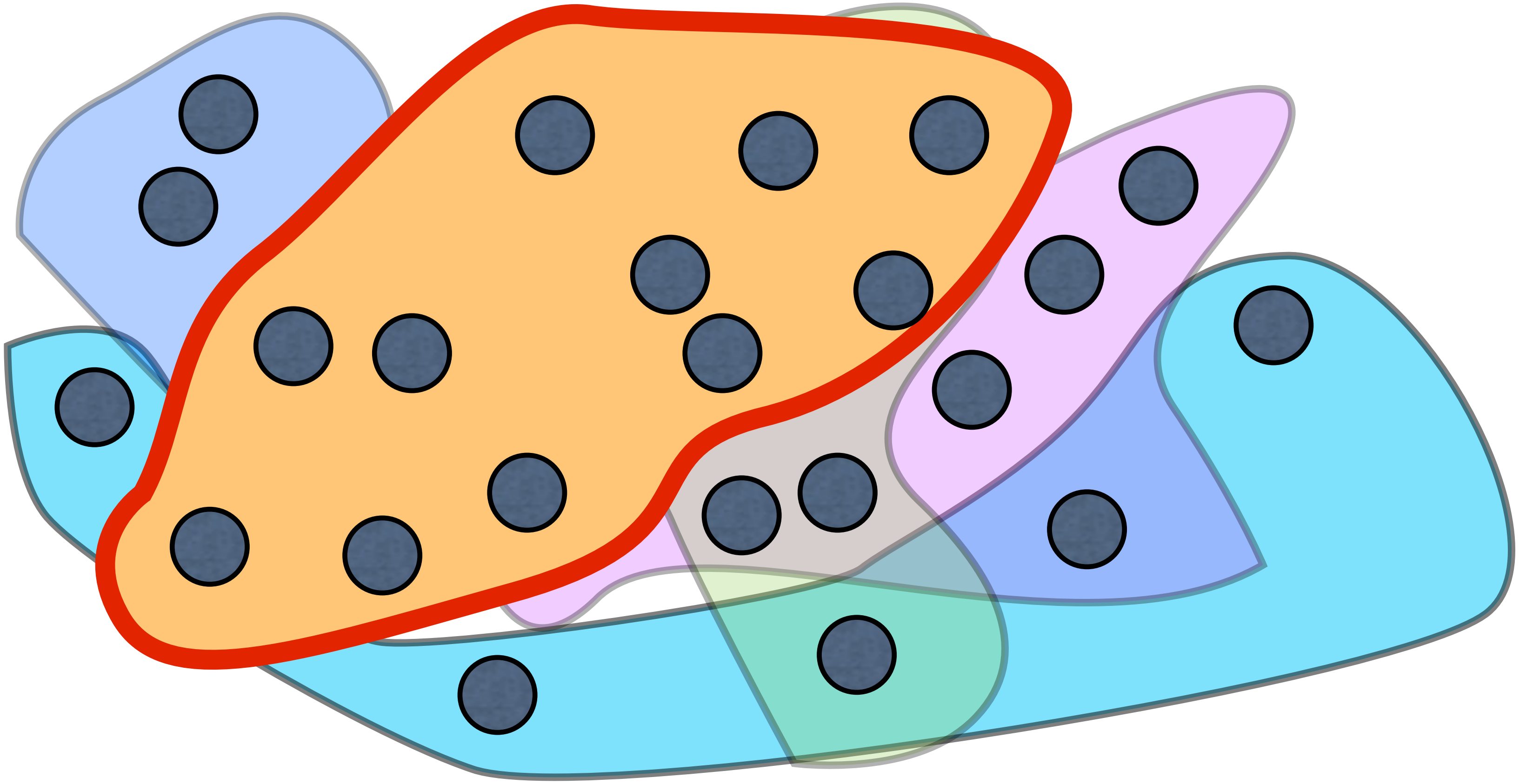

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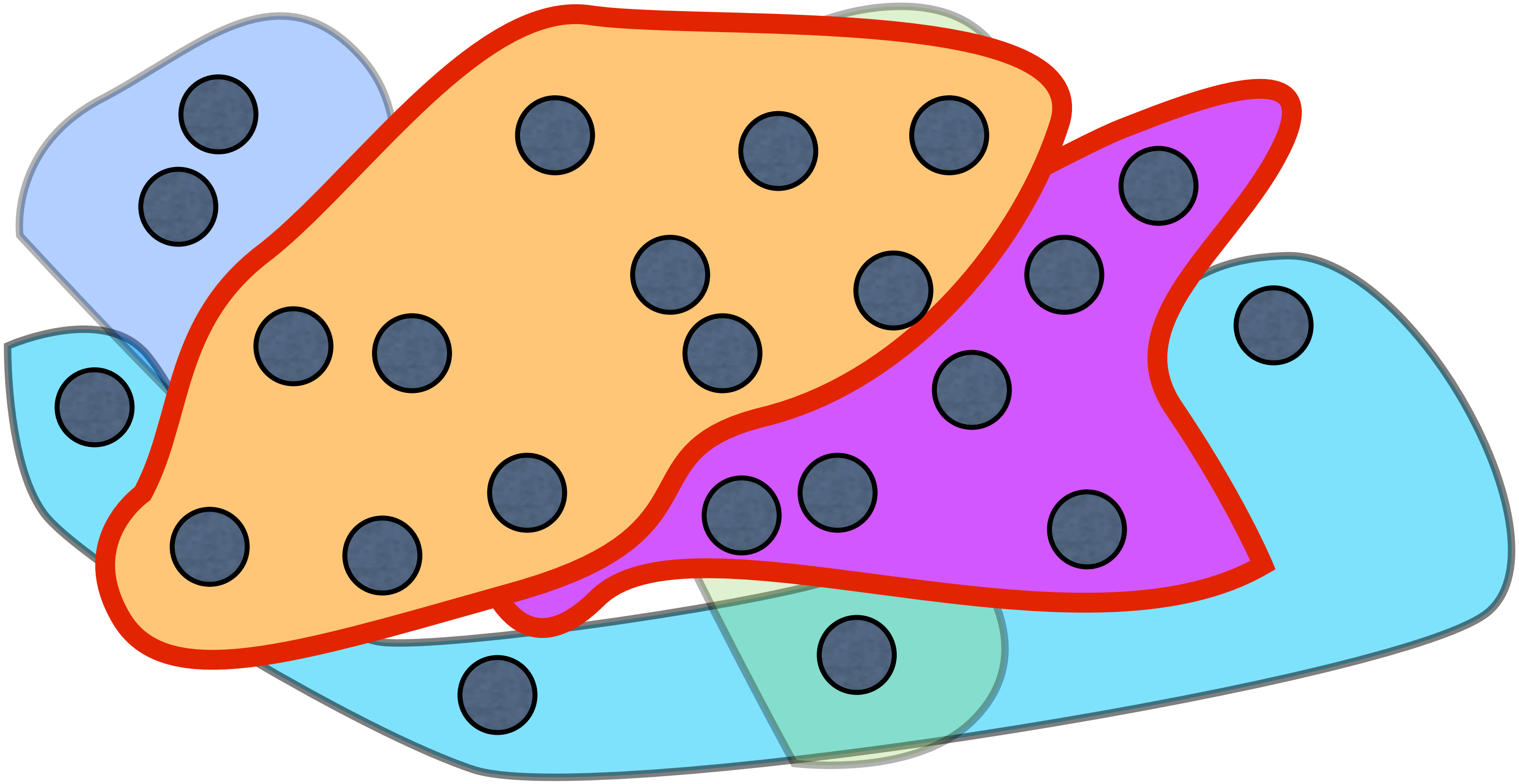
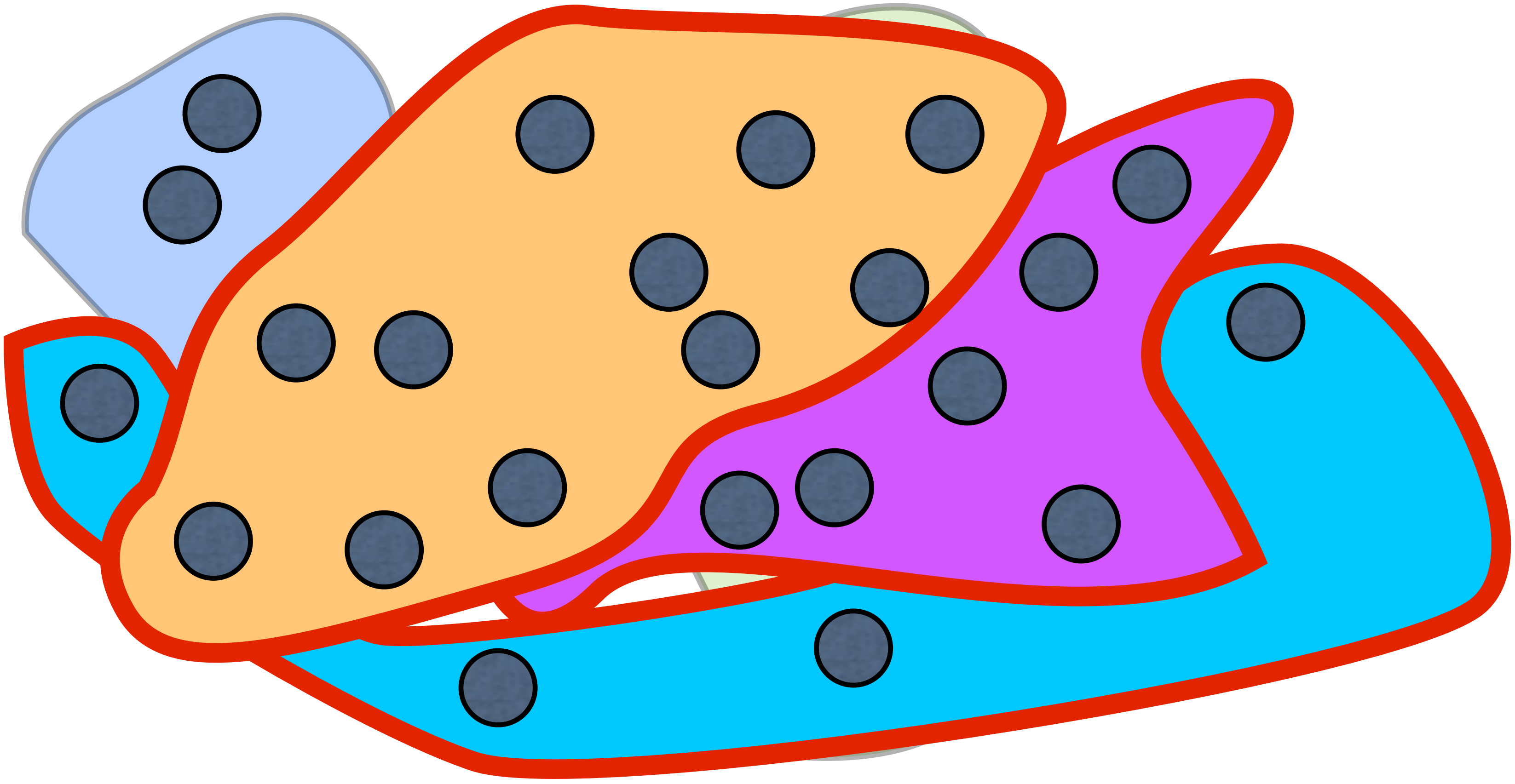


A Greedy Algorithm

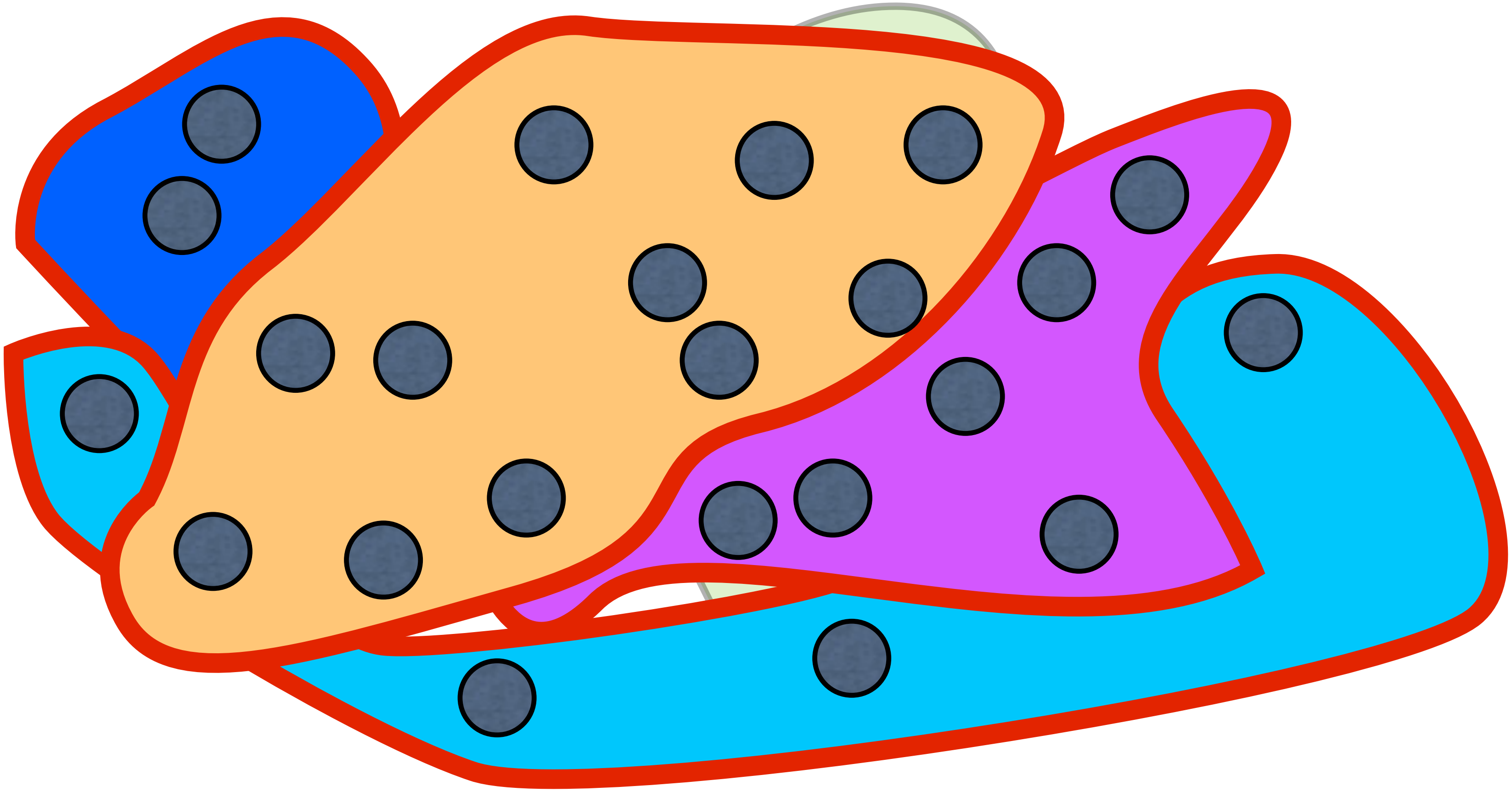
Strategy: Pick the set that maximizes # new elements covered



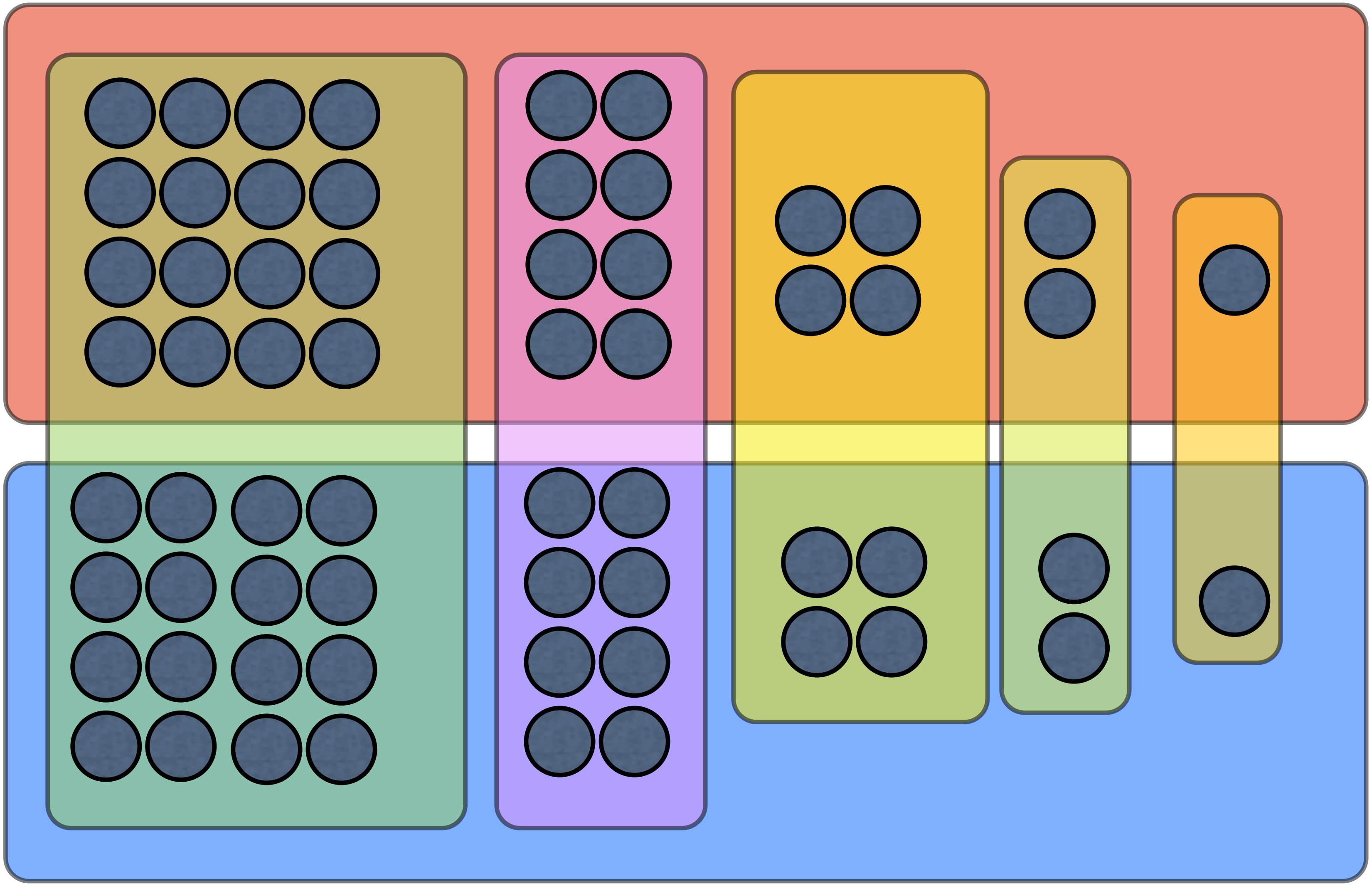
A Greedy Algorithm

Strategy: Pick the set that maximizes # new elements covered

Thm: Greedy has ln n approximation ratio



A Tight Example for Greedy



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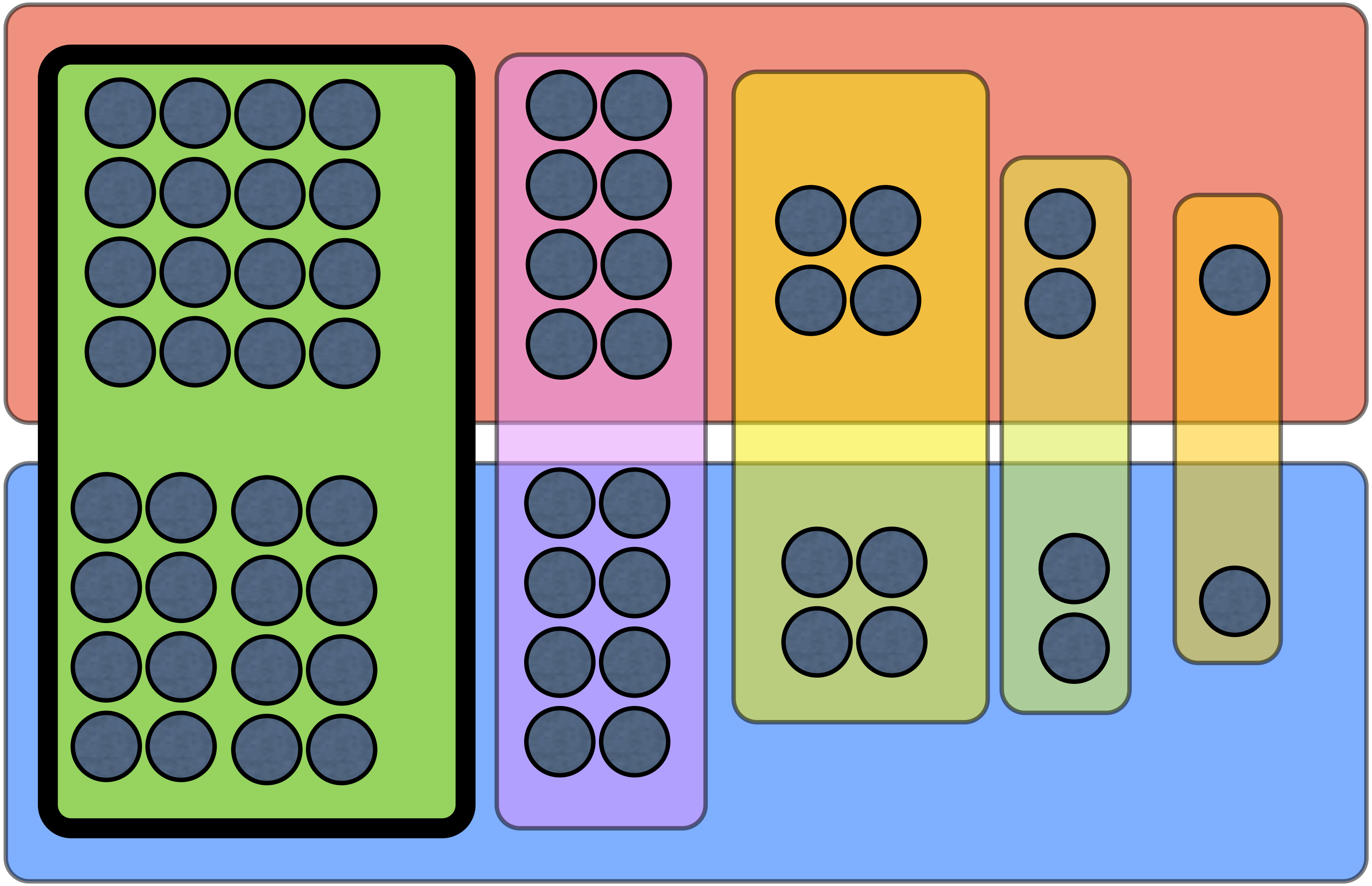

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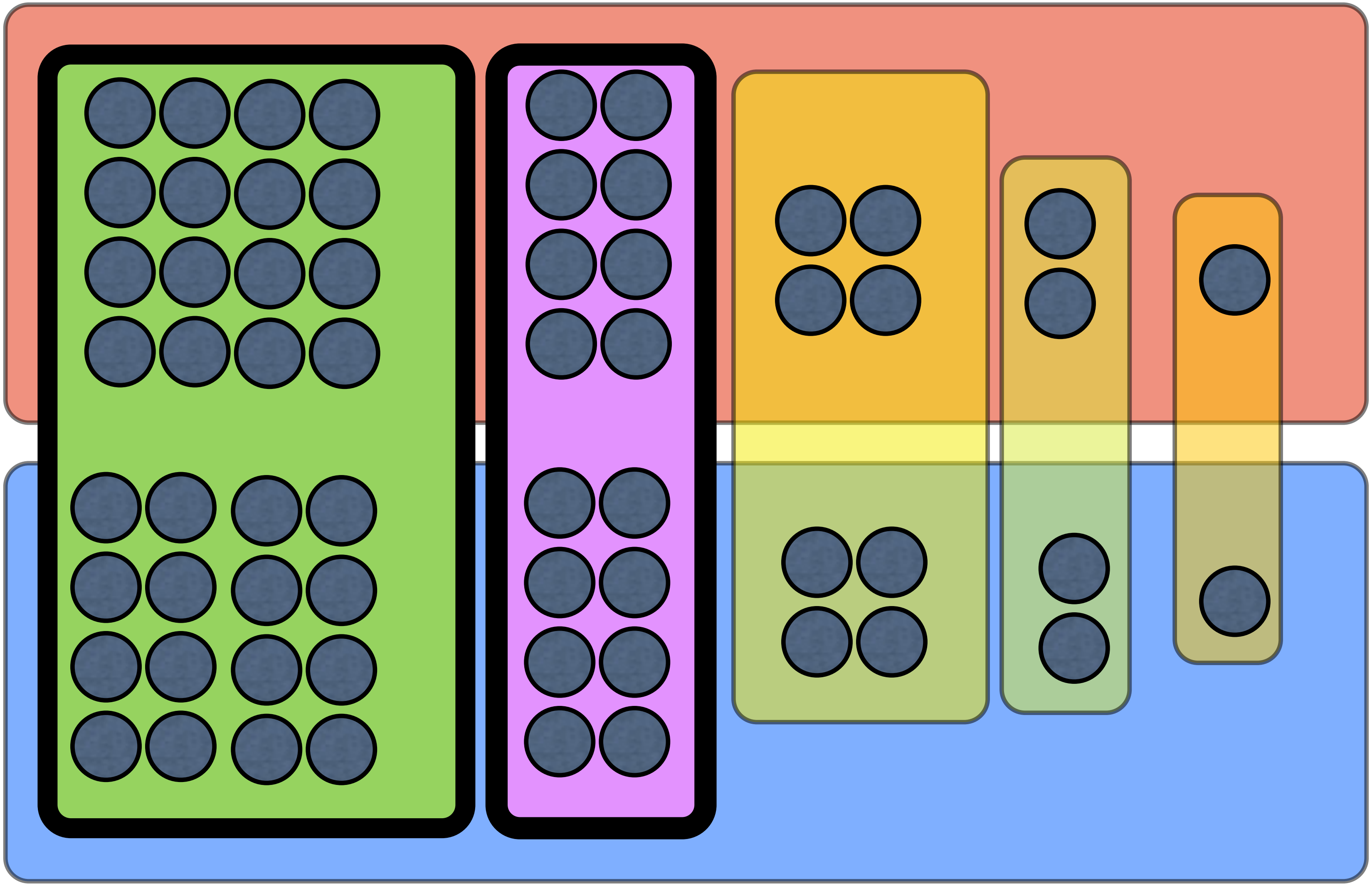

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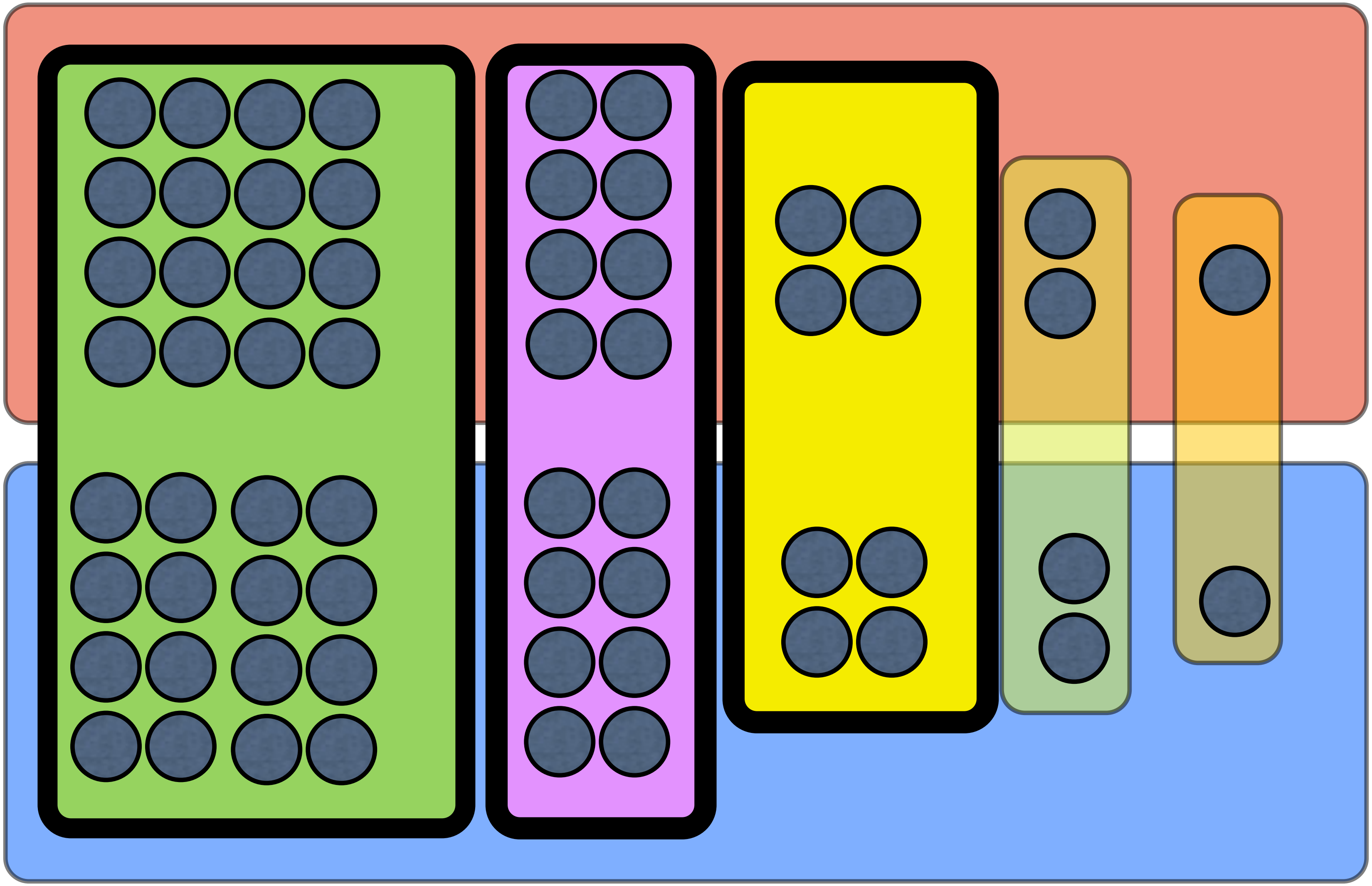

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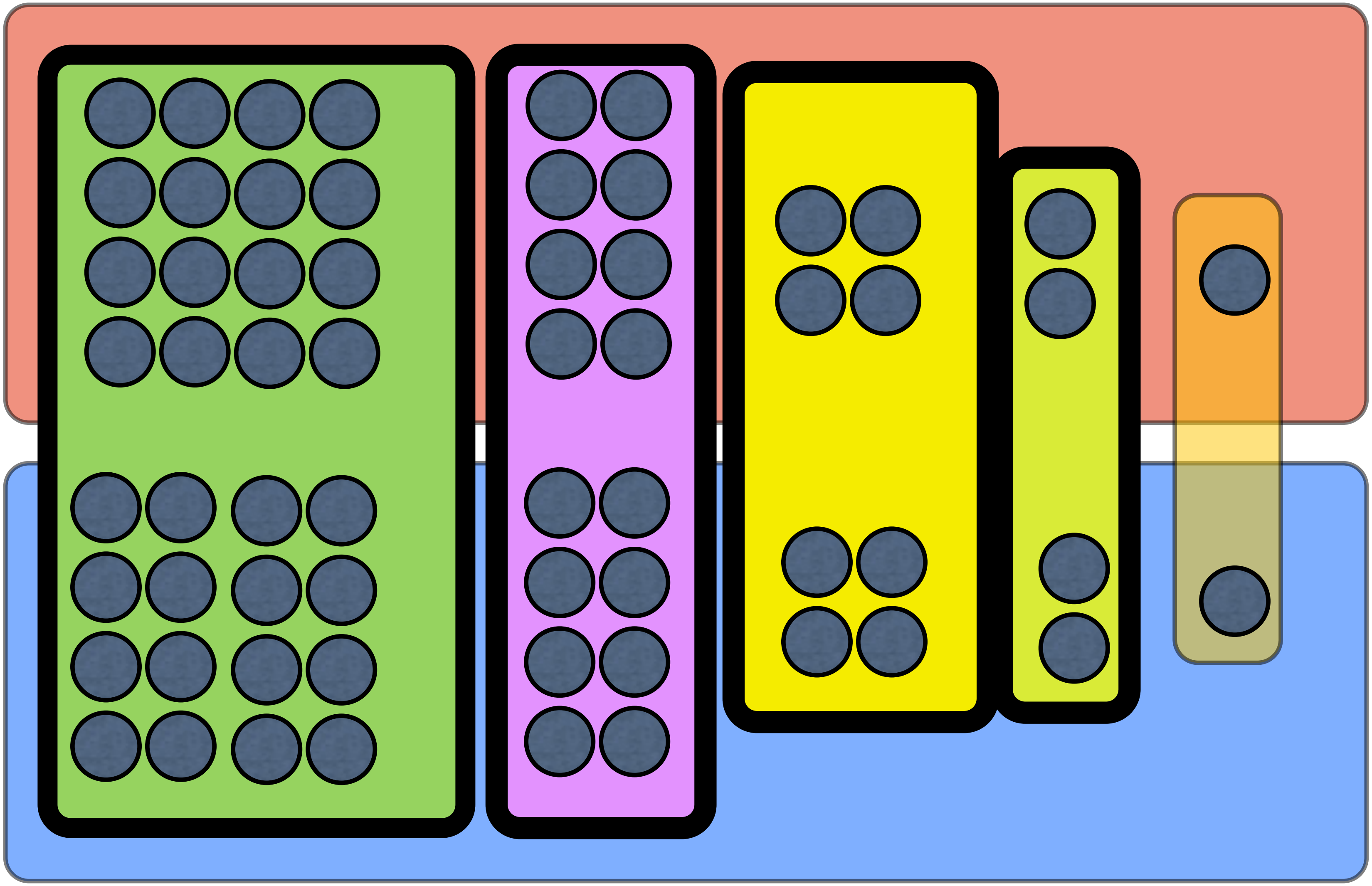

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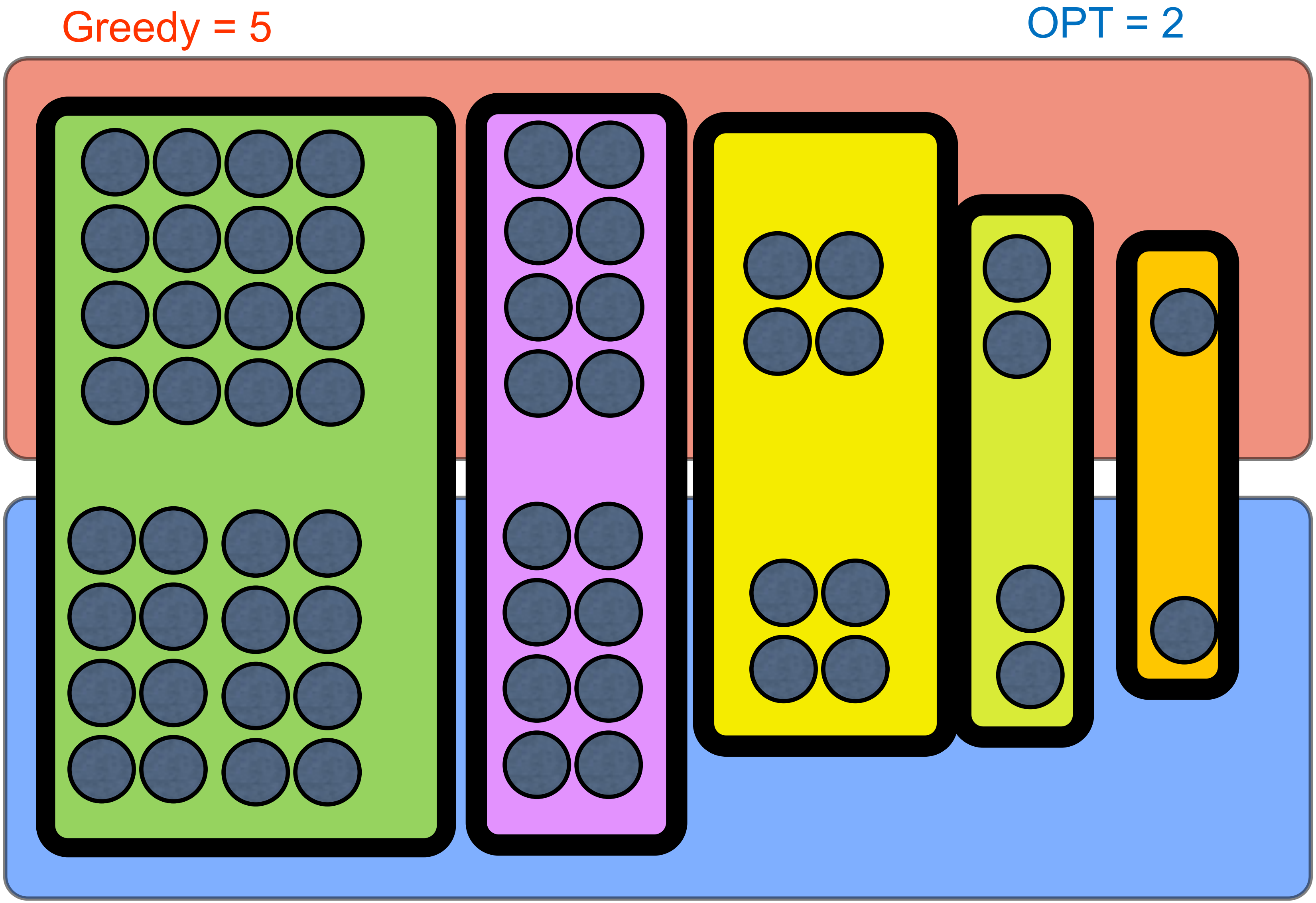


Greedy = 5

OPT = 2

Greedy Gives O(log(n)) approximation

Thm: If the best solution has k sets, greedy finds at most k ln(n) sets.

Pf: Suppose OPT=k

There is set that covers 1/k fraction of remaining elements, since there are k sets that cover all remaining elements.

So in each step, algorithm will cover 1/k fraction of remaining elements.

#elements uncovered after t steps

$$\leq n\left(1-\frac{1}{k}\right)^{t} \leq ne^{-\frac{t}{k}}$$

So after $t = k\ln n$ steps, # uncovered elements < 1.

Approximation Alg Summary

- To design approximation Alg, always find a way to lower bound OPT

- The best known approximation Alg for vertex cover is the greedy.
  - It has been open for 50 years to obtain a polynomial time algorithm with approximation ratio better than 2

- The best known approximation Alg for set cover is the greedy.
  - It is NP-Complete to obtain better than ln n approximation ratio for set cover.

Strengthening Induction Hypothesis

We have seen examples on how to design algorithms by induction

Basic Idea: A solution to every instance can be constructed from solutions of smaller instances

In some cases it may help to strengthen the IH.
High-level plan: Prove $P(n) \wedge Q(n)$ inductively.

IH: Assume $P(n-1) \wedge Q(n-1)$.

IS: You may use $Q(n-1)$ to help you to prove $P(n)$
Remember you also have to prove $Q(n)$.

Maximum Consecutive Subsequence

Problem: Given a sequence $x_1, \ldots, x_n$ of integers (not necessarily positive),

Goal: Find a subsequence of consecutive elements s.t., the sum of its numbers is maximum.

$$1 \quad -3 \quad \boxed{7 \quad -2 \quad -3 \quad 8} \quad -10 \quad 1 \quad -7$$

Applications: Figuring out the highest interest rate period in stock market

Brute Force Approach

Try all consecutive subsequences of the input sequence.

There are $\binom{n}{2} = \Theta(n^2)$ such sequences.

We can compute the sum of numbers in each such sequence in $O(n)$ steps.

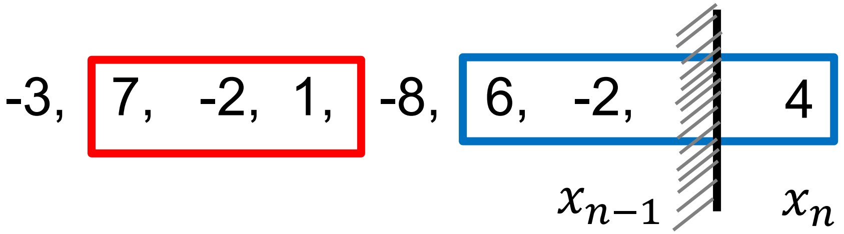
So, the ALG runs in $O(n^3)$.

With a clever loop we can do this in $O(n^2)$.
But, can we solve in linear time?

First Attempt (Induction)

Suppose we can find the maximum-sum subsequence of $x_1, \ldots, x_{n-1}$. Say it is $x_i, \ldots, x_j$

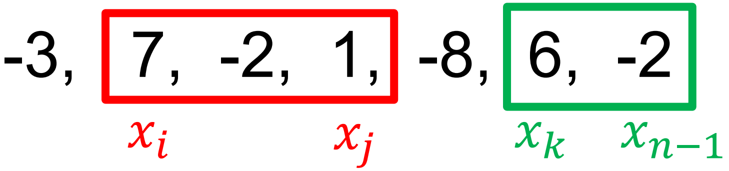
- If $x_n < 0$ then it does not belong to the largest subsequence. So, we can output $x_i, \ldots, x_j$
- Suppose $x_n > 0$.
  - If $j = n - 1$ then $x_i, \ldots, x_n$ is the maximum-sum subsequence.
  - If $j < n - 1$ there are two possibilities
    1) $x_i, \ldots, x_j$ is still the maximum-sum subsequence
    2) A sequence $x_k, \ldots, x_n$ is the maximum-sum subseqence



Second Attempt (Strengthing Ind Hyp)

Stronger Ind Hypothesis: Given $x_1, \dots, x_{n-1}$ we can compute the maximum-sum subsequence, and the maximum-sum suffix subsequence.

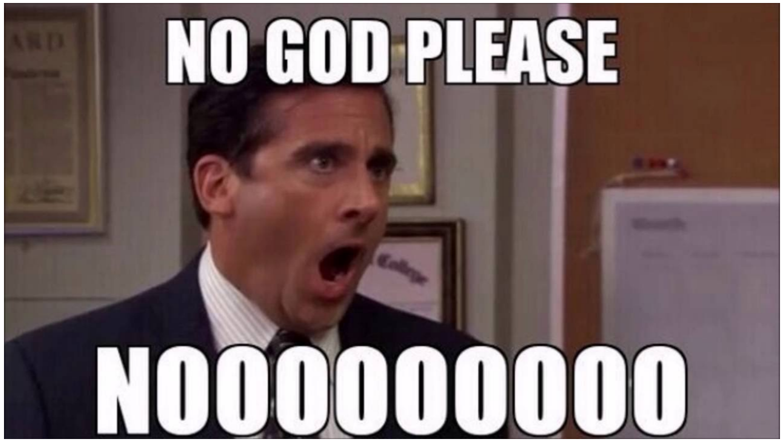
Can be empty



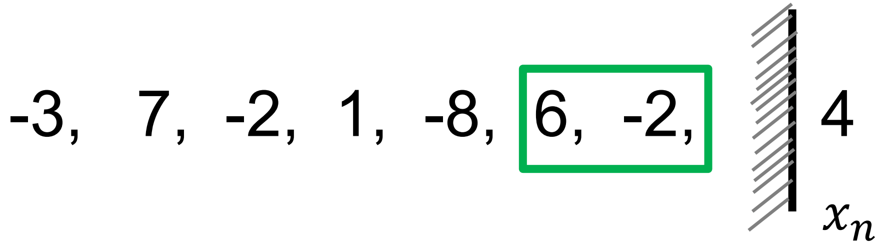
Say $\boldsymbol{x_i, \dots, x_j}$ is the maximum-sum and $x_k, \dots, x_{n-1}$ is the maximum-sum suffix subsequences.

- If $x_k + \cdots + x_{n-1} + x_n > x_i + \cdots + x_j$ then $x_k, \dots, x_n$ will be the new maximum-sum subsequence

Are we done?



Updating Max Suffix Subsequence


-3, 7, -2, 1, -8, 6, -2, 4

x_n

Say $x_k, \dots, x_{n-1}$ is the maximum-sum suffix subsequences of $x_1, \dots, x_{n-1}$.

- If $x_k + \cdots + x_n \geq 0$ then,
  $x_k, \dots, x_n$ is the new maximum-sum suffix subsequence
- Otherwise,
  The new maximum-sum suffix is the empty string.

Maximum Sum Subsequence ALG

```
Initialize S=0 (Sum of numbers in Maximum Subseq)
Initialize U=0 (Sum of numbers in Maximum Suffix)
for  (i=1 to n) {
    if (x[i] + U > S)
        S = x[i] + U

    if (x[i] + U > 0)
        U = x[i] + U
    else
        U = 0
}
Output S.
```

	-3	7	-2	1	-8	6	-2	4
S=0	0	7	7	7	7	7	7	8
U=0	0	7	5	6	0	6	4	8

Pf of Correct: Maximum Sum Subseq

Ind Hypo: Suppose

- $x_i, \dots, x_j$ is the max-sum-subseq of $x_1, \dots, x_{n-1}$
- $x_k, \dots, x_{n-1}$ is the max-suffix-sum-sub of $x_1, \dots, x_{n-1}$

Ind Step: Suppose $x_a, \dots, x_b$ is the max-sum-subseq of $x_1, \dots, x_n$

Case 1 ($b < n$): $x_a, \dots, x_b$ is also the max-sum-subseq of $x_1, \dots, x_{n-1}$
So, $a = i, b = j$ and the algorithm correctly outputs OPT

Case 2 ($b = n$): We must have $x_a, \dots, x_{b-1}$ is the max-suff-sum of $x_1, \dots, x_{n-1}$.
If not, then

$$x_k + \cdots x_{n-1} > x_a + \cdots + x_{n-1}$$

So, $x_k + \cdots + x_n > x_a + \cdots + x_b$ which is a contradiction.
Therefore, $a = k$ and the algorithm correctly outputs OPT

Special Cases (You don't need to mention if follows from above):

- The max-suffix-sum is empty string
- There are multiple maximum sum subsequences.

Pf of Correct: Max-Sum Suff Subseq

Ind Hypo: Suppose

- $x_i, \ldots, x_j$ is the max-sum-subseq of $x_1, \ldots, x_{n-1}$
- $x_k, \ldots, x_{n-1}$ is the max-suffix-sum-sub of $x_1, \ldots, x_{n-1}$

Ind Step: Suppose $x_a, \ldots, x_n$ is the max-sum-subseq of $x_1, \ldots, x_n$
Note that we may also have an empty sequence

Case 1 (OPT is empty): Then, we must have $x_k + \cdots + x_n < 0$. So the algorithm correctly finds max-suffix-sum subsequence.

Case 2 ($x_a, \ldots, x_n$ is nonempty): We must have $x_a + \cdots + x_n \geq 0$.
Also, $x_a, \ldots, x_{n-1}$ must be the max-suffix-sum of $x_1, \ldots, x_{n-1}$. If not,

$$x_a + \cdots + x_{n-1} < x_k + \cdots + x_{n-1}$$

which implies $x_a + \cdots + x_n < x_k + \cdots + x\_n$ which is a contradiction.

Therefore, $a = k$. So, the algorithm correctly finds max-suffix-sum subseqence.

Summary

- Try to reduce an instance of size n to smaller instances
  - Never solve a problem twice
- Before designing an algorithm study properties of optimum solution
- If ordinary induction fails, you may need to strengthen the induction hypothesis