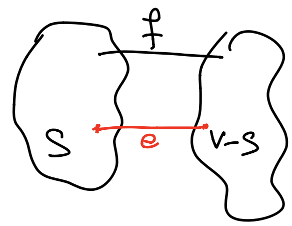
Lem. Fix a cut $(S, V-S)$. If e is the smallest edge in this cut, it must belong to all MSTs.



Pf. [By contradiction!].

$\exists\, T^*$ (MST) s.t. $e \notin T^*$.

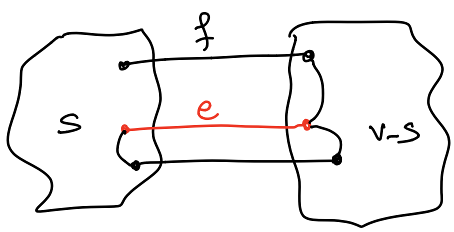
Supp $f \in T^*$ such that $f \in (S, V-S)$.

$\Rightarrow c_e < c_f$.

Swap e with f, $T' = T^* - f + e$ has smaller cost

We need to show T' is a spanning tree.

$n-1$ edges ✓



Incorrect!

Spanning Trees.
- connected
- acyclic
- $n-1$ edges

any two imply the third.

Add e to T^*. That makes a cycle C.

C crosses $(S, V-S)$ even # times $\Rightarrow \exists\, g \in C$ $(g \neq e)$, s.t. $g \in (S, V-S)$

From assumption on e, $c_e < c_g$. $T' = T^* - g + e$.

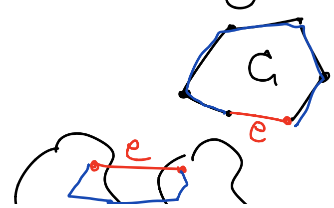
T' has smaller cost.

We need to show T' is a tree. $n-1$ edges ✓

Connectedness: $T^* + e$ is connected, so by removing an edge from cycle C $T' = T^* + e - g$ remains connected $\Rightarrow T'$ is a tree. A contradiction! $\square$.

Lem Let e be largest edge in a cycle C. $\Rightarrow e \notin T^*$ (for any T^* MST).

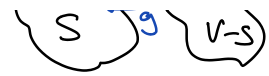
Pf. [By contradiction!]. Suppose $e \in T^*$.



Remove e from $T^* \Rightarrow$ we get disconnected

So $T^* - e$ has no edge in cut $(S, V-S)$

e is the unique edge of T^* in $(S, V-S)$



C crosses $(S, V-S)$ even # times $\Rightarrow \exists g \in C$ s.t. $g \neq e$ and $g \in (S, V-S)$.

Define $T' = T - e + g$. $c(T') < c(T)$ BC $c_e > c_g$.

We need to show T' is a tree.

$n-1$ edges ✓

T' is acyclic. $T^* - e$ acyclic.

if $T^* - e + g$ has a cycle g must be in it.

But that is not possible BC g is the unique edge in $(S, V-S)$

and any cycle containing g must cross this cut ≥ 2 time

So T' is a tree $\Rightarrow$ a contradiction! $\square$