CSE421: Design and Analysis of Algorithms April 11, 2019

Homework 2

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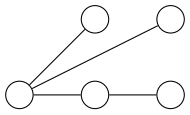
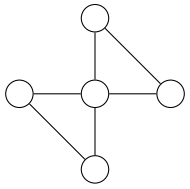
Please see https://courses.cs.washington.edu/courses/cse421/18wi/grading.html for general guidelines about Homework problems.

Most of the problems only require one or two key ideas for their solution. It will help you a lot to spell out these main ideas so that you can get most of the credit for a problem even if you err on the finer details. Please justify all answers.

P1) (20 points) Prove that for any tree T the number of leaves of T (i.e., vertices of degree 1) is at least the number of vertices of T of degree at least 3. For example, the tree in the following picture has exactly one node of degree at least 3 and 3 nodes of degree 1.



P2) Let G be a *connected* graph with n vertices and $n + k$ many edges. Show that G has at least $k + 1$ cycles. For example, the following graph has 5 vertices and 6 edges and 2 cycles.



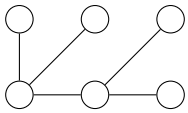
P3) (25 points) For a pair of nodes u, v in a graph $G = (V, E)$ we write $dist(u, v)$ to denote the length of the shortest path from u to v in G. The *diameter* of G is the maximum distance between any pair of nodes, i.e., $\max_{u,v} dist(u, v)$. In this exercise we design an $O(n)$ time algorithm to output the diameter of a tree.

a) (10 points) For a tree T, let u, v be a pair of vertices of T such that $dist(u, v)$ is equal to diameter of T. Prove that u, v are leaves of T.

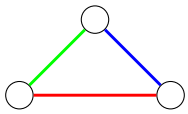
b) (10 points) Suppose we are given a tree T. We say a vertex v is special if v is one endpoint of the diameter of T, i.e., there exists a vertex u such that $dist(u, v)$ is the diameter of T. Design an $O(n)$-time algorithm that given tree T and a special vertex v, outputs the diameter of T. You don't need to output u.

c) (**Extra Credit**) Let a be an arbitrary vertex of T and let b be the farthest vertex from a in T. Prove that b is special.

d) (5 points) Design an $O(n)$-time algorithm to output the diameter of a tree. You just need to output the maximum value of $dist(u, v)$ over all u, v, e.g., for the following tree you should output 3



P4) (20 points) Given a graph $G = (V, E)$ with n vertices such that the degree of every vertex of G is at most k. Show that we can color the edges of G with at most $2k - 1$ colors such that any pair of edges e, f which are incident to the same vertex have distinct colors. For example, if G is a triangle, we have $k = 2$, and we can color edges of G with $2k - 1 = 3$ colors as follows:



P5) **Extra Credit:** Prove that we can color the edges of every graph G with two colors (red and blue) such that, for every vertex v, the number of red edges touching v and the number of blue edges touch v differ by at most 2.