CSE 421
Algorithms
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Lecture 26
NP-Completeness
NP Completeness

I can't find an efficient algorithm, I guess I'm just too dumb.

I can't find an efficient algorithm, but neither can all these famous people.
Algorithms vs. Lower bounds

• Algorithmic Theory
  – What we can compute
    • I can solve problem X with resources R
  – Proofs are almost always to give an algorithm that meets the resource bounds

• Lower bounds
  – How do we show that something can’t be done?
Theory of NP Completeness
The Universe

NP-Complete

NP

P
Polynomial Time

• P: Class of problems that can be solved in polynomial time
  – Corresponds with problems that can be solved efficiently in practice
  – Right class to work with “theoretically”
Decision Problems

- Theory developed in terms of yes/no problems
  - Independent set
    - Given a graph G and an integer K, does G have an independent set of size at least K
  - Network Flow
    - Given a graph G with edge capacities, a source vertex s, and sink vertex t, and an integer K, does the graph have flow function with value at least K
# Definition of P

Decision problems for which there is a polynomial time algorithm

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
<th>Algorithm</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>MULTIPLE</td>
<td>Is $x$ a multiple of $y$?</td>
<td>Grade school division</td>
<td>51, 17</td>
<td>51, 16</td>
</tr>
<tr>
<td>RELPRIME</td>
<td>Are $x$ and $y$ relatively prime?</td>
<td>Euclid’s algorithm</td>
<td>34, 39</td>
<td>34, 51</td>
</tr>
<tr>
<td>EDIT-DISTANCE</td>
<td>Is the edit distance between $x$ and $y$ less than 5?</td>
<td>Dynamic programming</td>
<td>neither</td>
<td>acggt tttta</td>
</tr>
<tr>
<td>LSOLVE</td>
<td>Is there a vector $x$ that satisfies $Ax = b$?</td>
<td>Gaussian elimination</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
What is NP?

• Problems solvable in non-deterministic polynomial time . . .

• Problems where “yes” instances have polynomial time checkable certificates
Certificate examples

• Independent set of size K
  – The Independent Set

• Satisfiable formula
  – Truth assignment to the variables

• Hamiltonian Circuit Problem
  – A cycle including all of the vertices

• K-coloring a graph
  – Assignment of colors to the vertices
Certifiers and Certificates: 3-Satisfiability

SAT: Does a given CNF formula have a satisfying formula

Certificate: An assignment of truth values to the n boolean variables

Certifier: Check that each clause has at least one true literal,

instance s

\[
\left( \overline{x_1} \lor x_2 \lor x_3 \right) \land \left( x_1 \lor \overline{x_2} \lor x_3 \right) \land \left( x_1 \lor x_2 \lor x_4 \right) \land \left( x_1 \lor \overline{x_3} \lor \overline{x_4} \right)
\]

certificate t

\[
x_1 = 1, \ x_2 = 1, \ x_3 = 0, \ x_4 = 1
\]
Certifiers and Certificates: Hamiltonian Cycle

HAM-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a simple cycle $C$ that visits every node?

Certificate. A permutation of the $n$ nodes.

Certifier. Check that the permutation contains each node in $V$ exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.
Polynomial time reductions

• Y is Polynomial Time Reducible to X
  – Solve problem Y with a polynomial number of computation steps and a polynomial number of calls to a black box that solves X
  – Notations: $Y \leq_P X$
Lemmas

• Suppose \( Y \prec_p X \). If \( X \) can be solved in polynomial time, then \( Y \) can be solved in polynomial time.

• Suppose \( Y \prec_p X \). If \( Y \) cannot be solved in polynomial time, then \( X \) cannot be solved in polynomial time.
NP-Completeness

• A problem X is NP-complete if
  – X is in NP
  – For every Y in NP, \( Y \preceq_p X \)

• X is a “hardest” problem in NP

• If X is NP-Complete, Z is in NP and X \( \preceq_p Z \)
  – Then Z is NP-Complete
Cook’s Theorem

• The Circuit Satisfiability Problem is NP-Complete
Circuit SAT

Find a satisfying assignment
History

• Jack Edmonds
  – Identified NP

• Steve Cook
  – Cook’s Theorem – NP-Completeness

• Dick Karp
  – Identified “standard” collection of NP-Complete Problems

• Leonid Levin
  – Independent discovery of NP-Completeness in USSR
P vs. NP Question

NP-Complete

NP

P

P

NP
Populating the NP-Completeness Universe

• Circuit Sat $\leq_p$ 3-SAT
• 3-SAT $\leq_p$ Independent Set
• 3-SAT $\leq_p$ Vertex Cover
• Independent Set $\leq_p$ Clique
• 3-SAT $\leq_p$ Hamiltonian Circuit
• Hamiltonian Circuit $\leq_p$ Traveling Salesman
• 3-SAT $\leq_p$ Integer Linear Programming
• 3-SAT $\leq_p$ Graph Coloring
• 3-SAT $\leq_p$ Subset Sum
• Subset Sum $\leq_p$ Scheduling with Release times and deadlines
Sample Problems

- Independent Set
  - Graph $G = (V, E)$, a subset $S$ of the vertices is independent if there are no edges between vertices in $S$
Vertex Cover

- Graph $G = (V, E)$, a subset $S$ of the vertices is a vertex cover if every edge in $E$ has at least one endpoint in $S$
Cook’s Theorem

- The Circuit Satisfiability Problem is NP-Complete

- Circuit Satisfiability
  - Given a boolean circuit, determine if there is an assignment of boolean values to the input to make the output true
Circuit SAT

Find a satisfying assignment
Proof of Cook’s Theorem

• Reduce an arbitrary problem Y in NP to X
• Let A be a non-deterministic polynomial time algorithm for Y
• Convert A to a circuit, so that Y is a Yes instance iff and only if the circuit is satisfiable