CSE 421 Algorithms

Autumn 2019, Lecture 20

Memory Efficient Dynamic Programming

Longest Common Subsequence

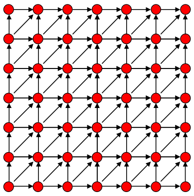
- C=$c_1 \ldots c_g$ is a subsequence of A=$a_1 \ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

LCS(BARTHOLEMEWSIMPSON, KRUSTYTHECLOWN)
= RTHOWN

LCS Optimization

- A = $a_1a_2 \ldots a_m$, B = $b_1b_2 \ldots b_n$
- Opt[j, k] is the length of LCS($a_1a_2 \ldots a_j$, $b_1b_2 \ldots b_k$)
- Optimization Recurrence:
  - If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]
  - If a_j != b_k, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

Dynamic Programming Computation

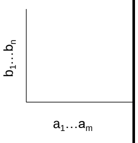


Code to compute Opt[n, m]

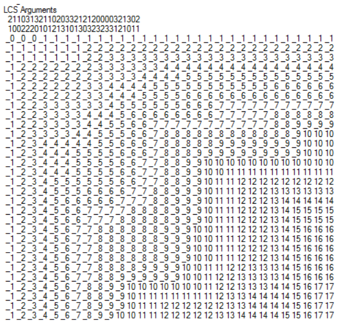
```
for (int i = 0; i < n; i++)
   for (int j = 0; j < m; j++)
      if (A[ i ] == B[ j ] )
         Opt[ i,j ] = Opt[ i-1, j-1 ] + 1;
      else if (Opt[ i-1, j ] >= Opt[ i, j-1 ])
          Opt[ i, j ] := Opt[ i-1, j ];
      else
          Opt[ i, j ] := Opt[ i, j-1];
```

Storing the path information

```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
      for j := 1 to n
            if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
            else if Opt[i-1, j] >= Opt[i, j-1]
                  {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
            else       {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```



Reconstructing Path from Distances



```
LCS_Arguments
 211031321102033212120000321302
 100222010121310130323233121011
_0 _0 _0 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1
_1 _1 _1 _1 _1 _1 _1 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2
_1 _1 _1 _1 _1 _1 _1 _2 _2 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _3 _3 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6
_1 _2 _2 _3 _3 _3 _3 _3 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7
_1 _2 _2 _3 _3 _3 _3 _4 _4 _4 _5 _6 _6 _6 _6 _6 _6 _6 _6 _7 _7 _7 _7 _7 _8 _8 _8 _8 _8 _8
_1 _2 _2 _3 _3 _3 _3 _4 _4 _5 _5 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _8 _8 _9 _9 _9 _9
_1 _2 _3 _3 _3 _3 _4 _4 _5 _5 _5 _6 _6 _7 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _9 10 10 10
_1 _2 _3 _4 _4 _4 _4 _4 _5 _5 _6 _6 _6 _7 _8 _8 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _7 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 10 10 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 11 11 11 11 11 11 11 11 11 11
_1 _2 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 11 11 12 12 12 12 12 12 12 12 12
_1 _2 _3 _4 _5 _5 _5 _6 _6 _6 _6 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 13 13 13 13 13
_1 _2 _3 _4 _5 _6 _6 _6 _6 _6 _7 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 14 14 14 14
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _7 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _8 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _9 _9 _9 _9 10 10 11 11 12 12 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _9 _9 _9 _9 _9 _9 10 10 11 12 12 13 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 11 12 12 13 13 13 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 11 11 11 11 11 11 11 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 _9 10 11 11 12 12 12 12 12 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 10 10 11 11 12 12 12 12 12 13 13 14 14 14 14 15 15 16 17 17
```

How good is this algorithm?

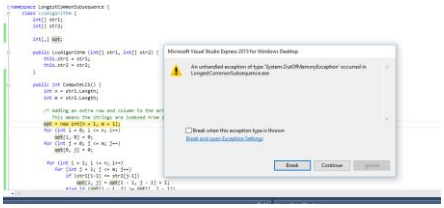
- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.

Implementation 1

```
public int ComputeLCS() {
            int n = str1.Length;
            int m = str2.Length;

            int[,] opt = new int[n + 1, m + 1];
            for (int i = 0; i <= n; i++)
                opt[i, 0] = 0;
            for (int j = 0; j <= m; j++)
                opt[0, j] = 0;

             for (int i = 1; i <= n; i++)
                for (int j = 1; j <= m; j++)
                    if (str1[i-1] == str2[j-1])
                        opt[i, j] = opt[i - 1, j - 1] + 1;
                    else if (opt[i - 1, j] >= opt[i, j - 1])
                        opt[i, j] = opt[i - 1, j];
                    else
                        opt[i, j] = opt[i, j - 1];

            return opt[n,m];
        }
```

N = 17000

Runtime should be about 5 seconds*



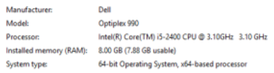
```
namespace LongestCommonSubsequence {
    class LcsAlgorithm {
        int[] str1;
        int[] str2;

        int[,] opt;

        public LcsAlgorithm (int[] str1, int[] str2) {
            this.str1 = str1;
            this.str2 = str2;
        }

        public int ComputeLCS() {
            int n = str1.Length;
            int m = str2.Length;

            /* Adding an extra row and column to the ar
               This means the strings are indexed from
            opt = new int[n + 1, m + 1];
            for (int i = 0; i <= n; i++)
                opt[i, 0] = 0;
            for (int j = 0; j <= m; j++)
                opt[0, j] = 0;

             for (int i = 1; i <= n; i++)
                for (int j = 1; j <= m; j++)
                    if (str1[i-1] == str2[j-1])
                        opt[i, j] = opt[i - 1, j - 1] + 1;
```

Microsoft Visual Studio Express 2015 for Windows Desktop

An unhandled exception of type 'System.OutOfMemoryException' occurred in LongestCommonSubsequence.exe

☐ Break when this exception type is thrown

* Personal PC, 6 years old



Manufacturer: Dell
Model: Optiplex 990
Processor: Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz 3.10 GHz
Installed memory (RAM): 8.00 GB (7.88 GB usable)
System type: 64-bit Operating System, x64-based processor

Implementation 2

```
public int SpaceEfficientLCS() {
            int n = str1.Length;
            int m = str2.Length;
            int[] prevRow = new int[m + 1];
            int[] currRow = new int[m + 1];

            for (int j = 0; j <= m; j++)
                prevRow[j] = 0;

            for (int i = 1; i <= n; i++) {
                currRow[0] = 0;
                for (int j = 1; j <= m; j++) {
                    if (str1[i - 1] == str2[j - 1])
                        currRow[j] = prevRow[j - 1] + 1;
                    else if (prevRow[j] >= currRow[j - 1])
                        currRow[j] = prevRow[j];
                    else
                        currRow[j] = currRow[j - 1];
                }
                for (int j = 1; j <= m; j++)
                    prevRow[j] = currRow[j];
            }

            return currRow[m];
        }
```

N = 300000

```
N: 10000 Base 2 Length: 8096   Gamma:  0.8096      Runtime:00:00:01.86
N: 20000 Base 2 Length: 16231  Gamma:  0.81155     Runtime:00:00:07.45
N: 30000 Base 2 Length: 24317  Gamma:  0.8105667   Runtime:00:00:16.82
N: 40000 Base 2 Length: 32510  Gamma:  0.81275     Runtime:00:00:29.84
N: 50000 Base 2 Length: 40563  Gamma:  0.81126     Runtime:00:00:46.78
N: 60000 Base 2 Length: 48700  Gamma:  0.8116667   Runtime:00:01:08.06
N: 70000 Base 2 Length: 56824  Gamma:  0.8117715   Runtime:00:01:33.36
```

```
N: 300000 Base 2 Length: 243605  Gamma:  0.8120167  Runtime:00:28:07.32
```

Observations about the Algorithm

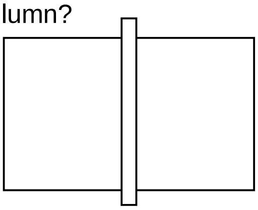
- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values
- The algorithm can be run from either end of the strings

Computing LCS in O(nm) time and O(n+m) space

- Divide and conquer algorithm
- Recomputing values used to save space

Divide and Conquer Algorithm

- Where does the best path cross the middle column?



- For a fixed i, and for each j, compute the LCS that has a_i matched with b_j

Constrained LCS

- $LCS_{i,j}(A,B)$: The LCS such that
  - $a_1,\ldots,a_i$ paired with elements of $b_1,\ldots,b_j$
  - $a_{i+1},\ldots a_m$ paired with elements of $b_{j+1},\ldots,b_n$
- $LCS_{4,3}$(abbacbb, cbbaa)

A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B),\ldots,LCS_{5,9}(A,B)$

A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B),\ldots,LCS_{5,9}(A,B)$

j	left	right
0	0	4
1	1	4
2	1	3
3	2	3
4	3	3
5	3	2
6	3	2
7	3	1
8	4	1
9	4	0

Computing the middle column

- From the left, compute LCS($a_1 \ldots a_{m/2}$,$b_1 \ldots b_j$)
- From the right, compute LCS($a_{m/2+1} \ldots a_m$,$b_{j+1} \ldots b_n$)
- Add values for corresponding j's

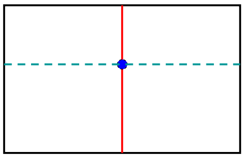
- Note – this is space efficient

Divide and Conquer

- A = $a_1, \ldots, a_m$ B = $b_1, \ldots, b_n$
- Find j such that
  - LCS($a_1 \ldots a_{m/2}$, $b_1 \ldots b_j$) and
  - LCS($a_{m/2+1} \ldots a_m$,$b_{j+1} \ldots b_n$) yield optimal solution
- Recurse

Algorithm Analysis

- T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm



Prove by induction that T(m,n) <= 2cmn

T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm

Memory Efficient LCS Summary

- We can afford O(nm) time, but we can't afford O(nm) space
- If we only want to compute the length of the LCS, we can easily reduce space to O(n+m)
- Avoid storing the value by recomputing values
  - Divide and conquer used to reduce problem sizes