CSE 421 Algorithms

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Lecture 19

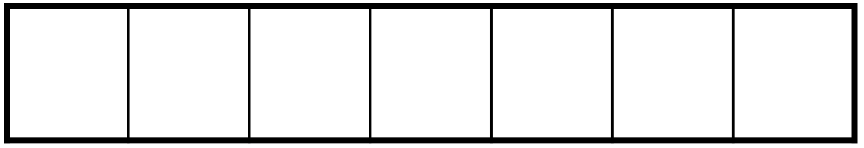
Dynamic Programming

Announcements

- Nov 11, No class (holiday)

One dimensional dynamic programming: Interval scheduling

$$Opt[\ j\ ] = \max\ (Opt[\ j - 1],\ w_j + Opt[\ p[\ j\ ]\ ])$$



Two dimensional dynamic programming

K-segment linear approximation

$Opt_k[\,j\,] = \min_i \{ Opt_{k-1}[\,i\,] + E_{i,j} \}$ for $0 < i < j$

4																	
3																	
2																	
1																	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Two dimensional dynamic programming

Subset sum and knapsack

$$Opt[\,j, K] = \max(Opt[\,j-1, K], Opt[\,j-1, K-w_j] + w_j)$$

$$Opt[\,j, K] = \max(Opt[\,j-1, K], Opt[\,j-1, K-w_j] + v_j)$$

4																	
3																	
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1																	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Alternate approach for Subset Sum

- Alternate formulation of Subset Sum dynamic programming algorithm
  - Sum[i, K] = true if there is a subset of {w_1,…w_i} that sums to exactly K, false otherwise
  - Sum [i, K] = Sum [i -1, K] **OR** Sum[i - 1, K - w_i]
  - Sum [0, 0] = true; Sum[i, 0] = false for i != 0

- To allow for negative numbers, we need to fill in the array between K_{min} and K_{max}

Run time for Subset Sum

- With n items and target sum K, the run time is O(nK)
- If K is 1,000,000,000,000,000,000,000,000 this is very slow
- Alternate brute force algorithm: examine all subsets: $O(n2^n)$

Optimal line breaking

Element distinctness has been a particular focus of lower bound analysis. The first time-space tradeoff lower bounds for the problem apply to structured algorithms. Borodin et al. [13] gave a time-space tradeoff lower bound for computing ED on *comparison* branching programs of $T \in \Omega(n^{3/2}/S^{1/2})$ and, since $S \geq \log_2 n$, $T \in \Omega(n^{3/2}\sqrt{\log n}/S)$. Yao [32] improved this to a near-optimal $T \in \Omega(n^{2-\epsilon(n)}/S)$, where $\epsilon(n) = 5/(\ln n)^{1/2}$. Since these lower bounds apply to the average case for randomly ordered inputs, by Yao's lemma, they also apply to randomized comparison branching programs. These bounds also trivially apply to all frequency moments since, for $k \neq 1$, $ED(x) = n$ iff $F_k(x) = n$. This near-quadratic lower bound seemed to suggest that the complexity of ED and F_k should closely track that of sorting.

Optimal Line Breaking

- Words have length w_i, line length L
- Penalty related to white space or overflow of the line
  - Quadratic measure often used
- Pen(i, j): Penalty for putting w_i, w_{i+1},…,w_j on the same line
- Opt[k, m]: minimum penalty for ending line k with w_m

String approximation

- Given a string S, and a library of strings B = $\{b_1, \ldots b_m\}$, construct an approximation of the string S by using copies of strings in B.

B = {abab, bbbaaa, ccbb, ccaacc}

S = abaccbbbaabbccbbccaabab

Formal Model

- Strings from B assigned to non-overlapping positions of S
- Strings from B may be used multiple times
- Cost of δ for unmatched character in S
- Cost of γ for mismatched character in S
  - MisMatch(i, j) – number of mismatched characters of b_j, when aligned starting with position i in s.

Design a Dynamic Programming Algorithm for String Approximation

- Compute Opt[1], Opt[2], . . ., Opt[n]
- What is Opt[k]?

Target string $S = s_1s_2\ldots s_n$
Library of strings $B = \{b_{1,}\ldots,b_m\}$
MisMatch(i,j) = number of mismatched characters with b_j when aligned starting at position i of S.

Opt[k] = fun(Opt[0],…,Opt[k-1])

- How is the solution determined from sub problems?

Target string $S = s_1 s_2 \ldots s_n$
Library of strings $B = \{b_{1,} \ldots, b_m\}$
MisMatch(i,j) = number of mismatched characters with b_j when aligned starting at position i of S.

Solution

```
for i := 1 to n
        Opt[k] = Opt[k-1] + δ;
        for j := 1 to |B|
                p = i – len(b_j);
                Opt[k] = min(Opt[k],  Opt[p-1] + γ MisMatch(p, j));
```

Longest Common Subsequence

- $C=c_1\ldots c_g$ is a subsequence of $A=a_1\ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

ocurranec

occurrence

attacggct

tacgacca

Determine the LCS of the following strings

BARTHOLEMEWSIMPSON

KRUSTYTHECLOWN

String Alignment Problem

- Align sequences with gaps

```
CAT TGA  AT

CAGAT AGGA
```

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Note: the problem is often expressed as a minimization problem, with $\gamma_{xx} = 0$ and $\delta_x > 0$

LCS Optimization

- $A = a_1a_2\ldots a_m$
- $B = b_1b_2\ldots b_n$

- Opt[j, k] is the length of LCS($a_1a_2\ldots a_j$, $b_1b_2\ldots b_k$)

Optimization recurrence

If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]

If a_j != b_k, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

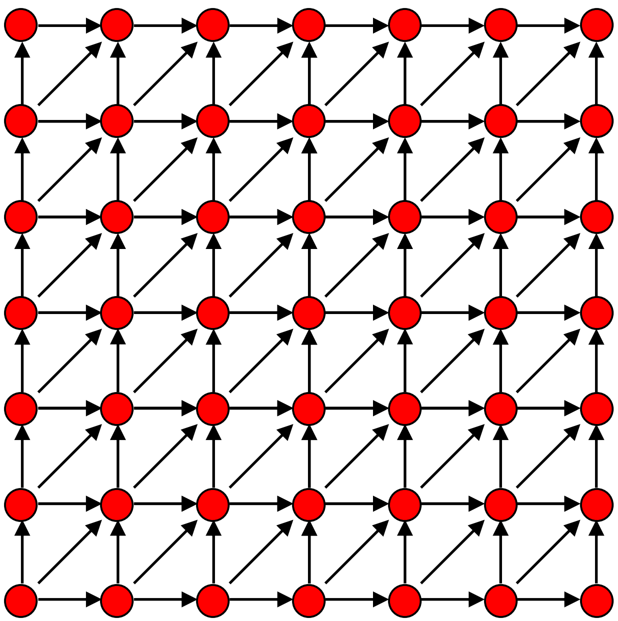
Give the Optimization Recurrence for the String Alignment Problem

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Opt[j, k] =

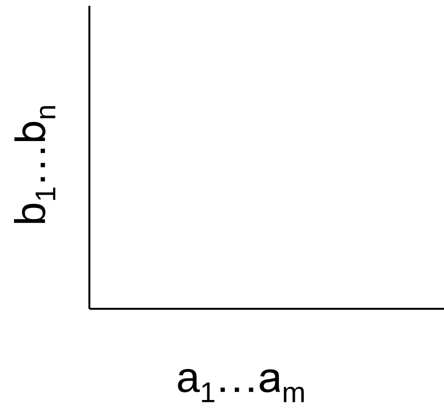
Let a_j = x and b_k = y
Express as minimization

Dynamic Programming Computation



Code to compute Opt[j,k]

Storing the path information




```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
        for j := 1 to n
                if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
                else if Opt[i-1, j] >= Opt[i, j-1]
                        {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
                else       {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```

How good is this algorithm?

- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.