CSE 421: Introduction to Algorithms

Fast Fourier Transform

Paul Beame

Integer Multiplication

- **Given:**
  - Two n-bit integers X and Y
    - $X = a_0 + a_1\,2 + a_2\,2^2 + \ldots + a_{n-2}2^{n-2} + a_{n-1}2^{n-1}$
    - $Y = b_0 + b_1\,2 + b_2\,2^2 + \ldots + b_{n-2}2^{n-2} + b_{n-1}2^{n-1}$
- **Compute:**
  - $2n-1$-bit integer $X\,Y$
  - $X\,Y = a_0b_0 + (a_0b_1+a_1b_0)\,2 + (a_0b_2+a_1b_1+a_2b_0)\,2^2 + \ldots + (a_{n-2}b_{n-1}+a_{n-1}b_{n-2})\,2^{2n-3} + a_{n-1}b_{n-1}\,2^{2n-2}$
- **Last time:** Karatsuba's Algorithm beats naïve algorithm, using $O(n^\alpha)$ where $\alpha = \log_2 3 = 1.59\ldots$

Polynomial Multiplication

- **Given:**
  - Degree $n-1$ polynomials P and Q
    - $P = a_0 + a_1\,x + a_2\,x^2 + \ldots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1}$
    - $Q = b_0 + b_1\,x + b_2\,x^2 + \ldots + b_{n-2}x^{n-2} + b_{n-1}x^{n-1}$
- **Compute:**
  - Degree $2n-2$ Polynomial $P\,Q$
  - $P\,Q = a_0b_0 + (a_0b_1+a_1b_0)\,x + (a_0b_2+a_1b_1+a_2b_0)\,x^2 + \ldots + (a_{n-2}b_{n-1}+a_{n-1}b_{n-2})\,x^{2n-3} + a_{n-1}b_{n-1}\,x^{2n-2}$
- **Obvious Algorithm, just like Integer Mult.:**
  - Compute all a_ib_j and collect terms
  - $\Theta(n^2)$ time

Divide and Conquer

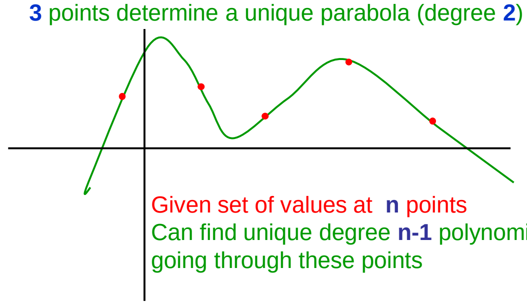
- Assume $n=2k$
  - $P = P_0 + P_1\,x^k$ where P_0 and P_1 are degree $k-1$ polys
  - Similarly $Q = Q_0 + Q_1\,x^k$
- $P\,Q = (P_0+P_1x^k)(Q_0+Q_1x^k)$
  $= P_0Q_0 + (P_1Q_0+P_0Q_1)x^k + P_1Q_1x^{2k}$
- **Naïve**: 4 sub-problems of size $k=n/2$ plus linear combining $T(n)=4\cdot T(n/2)+cn$ Solution $T(n) = \Theta(n^2)$
- **Karatsuba's** : 3 instead 4: $A \leftarrow P_0Q_0$ $B \leftarrow P_1Q_1$
  $C \leftarrow (P_0+P_1)(Q_0+Q_1)$ and then $C-A-B = P_1Q_0+P_0Q_1$
  so $T(n) = 3\,T(n/2) + cn$ and $T(n) = O(n^\alpha)$ where $\alpha = \log_2 3 = 1.59\ldots$

Integer and Polynomial Multiplication

- Naïve: $\Theta(n^2)$
- Karatsuba: $\Theta(n^{1.59\ldots})$
- Best known: $\Theta(n \log n)$
  - "Fast Fourier Transform"
  - FFT widely used for signal processing
  - Used in practice in symbolic manipulation systems like Maple
  - MUCH easier for Polynomial Multiplication than for integer multiplication because of ugly details with carries, etc.
    - Schonhage-Strassen (1971) gives $\Theta(n \log n \log\log n)$
    - Furer (2007) gives $\Theta(n \log n\ 2^{\log^* n})$
    - Harvey, van der Hoeven (2019) finally got $\Theta(n \log n)$

Hints towards FFT: Interpolation

2 points determine a unique line (degree 1)
3 points determine a unique parabola (degree 2)



Given set of values at n points
Can find unique degree $n-1$ polynomial going through these points

Multiplying Polynomials by Evaluation & Interpolation

- Any degree **n-1** polynomial $R(y)$ is determined by $R(y_0)$, … $R(y_{n-1})$ for any **n** distinct $y_0,\ldots,y_{n-1}$
- To compute **PQ** (assume degree at most **n/2-1**)
  - Evaluate $P(y_0),\ldots, P(y_{n-1})$
  - Evaluate $Q(y_0),\ldots,Q(y_{n-1})$
  - Multiply values $P(y_i)Q(y_i)$ for $i=0,\ldots,n-1$
  - Interpolate to recover **PQ**

Interpolation

- Given values of degree **n-1** polynomial **R** at **n** distinct points $y_0,\ldots,y_{n-1}$
  - $R(y_0),\ldots,R(y_{n-1})$
- Compute coefficients $c_0,\ldots,c_{n-1}$ such that
  - $R(x)=c_0+c_1x+c_2x^2+\ldots+c_{n-1}x^{n-1}$
- System of linear equations in $c_0,\ldots,c_{n-1}$

$$c_0 + c_1y_0+c_2y_0^2+\ldots+c_{n-1}y_0^{n-1}=R(y_0)$$
$$c_0 + c_1y_1+c_2y_1^2+\ldots+c_{n-1}y_1^{n-1}=R(y_1)$$
$$\ldots$$
$$c_0 + c_1y_{n-1}+c_2y_{n-1}^2+..+c_{n-1}y_{n-1}^{n-1}=R(y_{n-1})$$

known

unknown

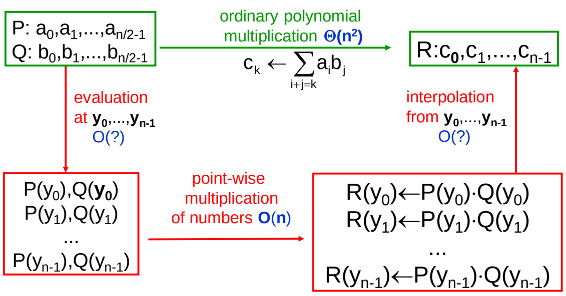
Interpolation: n equations in n unknowns

- Matrix form of the linear system

$$\begin{pmatrix} 1 & y_0 & y_0^2 & \ldots & y_0^{n-1} \\ 1 & y_1 & y_1^2 & \ldots & y_1^{n-1} \\ & \ldots & & & \\ & \ldots & & & \\ 1 & y_{n-1} & y_{n-1}^2 & \ldots & y_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} R(y_0) \\ R(y_1) \\ \cdot \\ \cdot \\ R(y_{n-1}) \end{pmatrix}$$

- Fact: Determinant of the matrix is $\prod_{i<j}(y_i-y_j)$ which is not **0** since points are distinct
  - System has a unique solution $c_0,\ldots,c_{n-1}$

Hints towards FFT: Evaluation & Interpolation



Karatsuba's algorithm and evaluation and interpolation

- Karatsuba's algorithm can be thought of as a way of multiplying two degree **1** polynomials (which have **2** coefficients) using only **3** multiplications
  - $PQ=(P_0+P_1z)(Q_0+Q_1z)$
    $= P_0Q_0 + (P_1Q_0+P_0Q_1)z + P_1Q_1z^2$
  - Evaluate at **0,1** plus compute P_1Q_1
    - $A = P(0)Q(0)= P_0Q_0$
    - $B = P_1Q_1$
    - $C = P(1)Q(1)=(P_0+P_1)(Q_0+Q_1)$
  - Alternative: replace **B** by the following: Evaluate at **-1**
    - $D = P(-1)Q(-1)=(P_0 -P_1)(Q_0 -Q_1)$
  - Interpolating, product is $A + (C-D)/2\ z + [(C+D)/2-A]\ z^2$

Evaluation at Special Points

- Evaluation of polynomial at **1** point takes **O(n)** time
  - So **2n** points (naively) takes $O(n^2)$—no savings
  - But the algorithm works no matter what the points are…
- So…choose points that are related to each other so that evaluation problems can share subproblems

The key idea: Evaluate at related points

- $P(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots+a_{n-1}x^{n-1}$
  $= a_0 + a_2x^2 + a_4x^4 + \ldots + a_{n-2}x^{n-2}$
  $+ a_1x+a_3x^3 + a_5x^5 + \ldots + a_{n-1}x^{n-1}$
  $= P_{even}(x^2) + x\, P_{odd}(x^2)$
- $P(-x)=a_0 - a_1x+a_2x^2 - a_3x^3+a_4x^4 - \ldots\ - a_{n-1}x^{n-1}$
  $= a_0 + a_2x^2 + a_4x^4 + \ldots + a_{n-2}x^{n-2}$
  $- (a_1x+a_3x^3 + a_5x^5 + \ldots + a_{n-1}x^{n-1})$
  $= P_{even}(x^2) - x\, P_{odd}(x^2)$

where $P_{even}(z) = a_0 + a_2z + a_4z^2 + \ldots + a_{n-2}z^{n/2-1}$

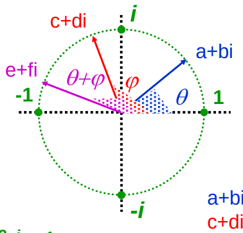
and $P_{odd}(z) = a_1+a_3z + a_5z^2 + \ldots + a_{n-1}z^{n/2-1}$

The key idea: Evaluate at related points

- So… if we have half the points as negatives of the other half
  - i.e., $y_{n/2}= -y_0$, $y_{n/2+1}= -y_1, \ldots, y_{n-1}= -y_{n/2-1}$

  then we can reduce the size n problem of evaluating degree $n-1$ polynomial P at n points to evaluating 2 degree $n/2 - 1$ polynomials P_{even} and P_{odd} at $n/2$ points $y_0^2, \ldots y_{n/2-1}^2$ and recombine answers with $O(1)$ extra work per point
- But to use this idea recursively we need half of $y_0^2, \ldots y_{n/2-1}^2$ to be negatives of the other half
  - If $y_{n/4}^2 = -y_0^2$, say, then $(y_{n/4}/y_0)^2 = -1$
  - Motivates use of complex numbers as evaluation points

Complex Numbers $i^2 = -1$



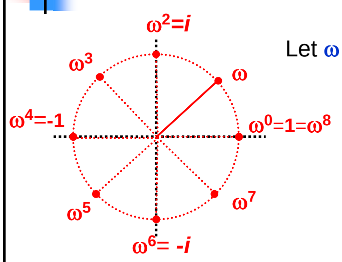
To multiply complex numbers:
1. add angles
2. multiply lengths
(all length 1 here)

e+fi = (a+bi)(c+di)

$a+bi = \cos\theta + i\sin\theta = e^{i\theta}$
$c+di = \cos\varphi + i\sin\varphi = e^{i\varphi}$
$e+fi = \cos(\theta+\varphi) + i\sin(\theta+\varphi) = e^{i(\theta+\varphi)}$

$e^{2\pi i} = 1$
$e^{\pi i} = -1$

Primitive n^{th} root of 1 $\omega=\omega_n= e^{i\,2\pi/n}$



Let $\omega = \omega_n = e^{i\,2\pi/n}$
$= \cos(2\pi/n) + i\sin(2\pi/n)$

$i^2 = -1$
$e^{2\pi i} = 1$

Facts about $\omega=e^{2\pi i/n}$ for even n

- $\omega = e^{2\pi i/n}$ for $i = \sqrt{-1}$
- $\omega^n = 1$
- $\omega^{n/2} = -1$
- $\omega^{n/2+j} = -\omega^j$ for all values of j
- $\omega^2 = e^{2\pi i/k}$ where $k=n/2$
- $\omega^j = \cos(2j\pi/n) + i\sin(2j\pi/n)$ so can compute with powers of ω
- ω^j is a root of $x^n-1= (x-1)(x^{n-1}+x^{n-2}+\ldots+1) =0$ but for $j\neq 0$, $\omega^j\neq 1$ so $\omega^{j(n-1)}+\omega^{j(n-2)} +\ldots+1=0$

The key idea for n even

- $P(\omega) = a_0+a_1\omega+a_2\omega^2+a_3\omega^3+a_4\omega^4+\ldots+a_{n-1}\omega^{n-1}$
  $= a_0 + a_2\omega^2 + a_4\omega^4 + \ldots + a_{n-2}\omega^{n-2}$
  $+ a_1\omega+a_3\omega^3 + a_5\omega^5 + \ldots + a_{n-1}\omega^{n-1}$
  $= P_{even}(\omega^2) + \omega\, P_{odd}(\omega^2)$
- $P(-\omega)=a_0 - a_1\omega+a_2\omega^2 - a_3\omega^3+a_4\omega^4 - \ldots\ - a_{n-1}\omega^{n-1}$
  $= a_0 + a_2\omega^2 + a_4\omega^4 + \ldots + a_{n-2}\omega^{n-2}$
  $- (a_1\omega+a_3\omega^3 + a_5\omega^5 + \ldots + a_{n-1}\omega^{n-1})$
  $= P_{even}(\omega^2) - \omega\, P_{odd}(\omega^2)$

where $P_{even}(x) = a_0 + a_2x + a_4x^2 + \ldots + a_{n-2}x^{n/2-1}$

and $P_{odd}(x) = a_1+a_3x + a_5x^2 + \ldots + a_{n-1}x^{n/2-1}$

The recursive idea for n a power of 2

- Goal:
  - Evaluate P at $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$
- Now
  - P_{even} and P_{odd} have degree $n/2-1$ where
  - $P(\omega^k) = P_{even}(\omega^{2k}) + \omega^k P_{odd}(\omega^{2k})$
  - $P(-\omega^k) = P_{even}(\omega^{2k}) - \omega^k P_{odd}(\omega^{2k})$
- Recursive Algorithm
  - Evaluate P_{even} at $1, \omega^2, \omega^4, \ldots, \omega^{n-2}$
  - Evaluate P_{odd} at $1, \omega^2, \omega^4, \ldots, \omega^{n-2}$
  - Combine to compute P at $1, \omega, \omega^2, \ldots, \omega^{n/2-1}$
  - Combine to compute P at $-1, -\omega, -\omega^2, \ldots, -\omega^{n/2-1}$ (i.e. at $\omega^{n/2}, \omega^{n/2+1}, \omega^{n/2+2}, \ldots, \omega^{n-1}$)

ω^2 is $e^{2\pi i/k}$ where $k=n/2$ so problems are of same type but smaller size

Analysis and more

- Run-time
  - $T(n)=2 \cdot T(n/2)+cn$ so $T(n)=O(n \log n)$
- So much for evaluation ... what about interpolation?
  - Given
    - $r_0=R(1)$, $r_1=R(\omega)$, $r_2=R(\omega^2), \ldots, r_{n-1}=R(\omega^{n-1})$
  - Compute
    - $c_0, c_1, \ldots, c_{n-1}$ s.t. $R(x)=c_0+c_1x+\ldots+c_{n-1}x^{n-1}$

Interpolation ≈ Evaluation: strange but true

- Non-obvious fact:
  - If we define a new polynomial $S(x) = r_0 + r_1 x + r_2 x^2 + \ldots + r_{n-1} x^{n-1}$ where $r_0, r_1, \ldots, r_{n-1}$ are the evaluations of R at $1, \omega, \ldots, \omega^{n-1}$
  - Then $c_j=S(\omega^{-j})/n$ for $k=0,\ldots,n-1$
  - Relies on the fact the interpolation (inverse) matrix has ij entry $\omega^{-(ij)}/n$ instead of ω^{ij}
- So...
  - evaluate S at $1, \omega^{-1}, \omega^{-2}, \ldots, \omega^{-(n-1)}$ then divide each answer by n to get the $c_0, \ldots, c_{n-1}$
  - ω^{-1} behaves just like ω did so the same $O(n \log n)$ evaluation algorithm applies !

Why this is called the discrete Fourier transform

- Real Fourier series
  - Given a real valued function f defined on $[0,2\pi]$ the Fourier series for f is given by $f(x)=a_0+a_1\cos(x) + a_2\cos(2x) + \ldots + a_m\cos(mx) + \ldots$ where

$$a_m = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos(mx)\,dx$$

  - is the component of f of frequency m
  - In signal processing and data compression one ignores all but the components with large a_m and there aren't many since

Why this is called the discrete Fourier transform

- Complex Fourier series
  - Given a function f defined on $[0,2\pi]$ the complex Fourier series for f is given by $f(z)=b_0+b_1 e^{iz} + b_2 e^{2iz} + \ldots + b_m e^{miz} + \ldots$ where

$$b_m = \frac{1}{2\pi}\int_0^{2\pi} f(z)\, e^{-miz}\,dz$$

is the component of f of frequency m

  - If we **discretize** this integral using values at n equally spaced points between 0 and 2π we get

$2\pi/n$ apart

$$\bar{b}_m = \frac{1}{n}\sum_{k=0}^{n-1} f_k\, e^{-2kmi\pi/n} = \frac{1}{n}\sum_{k=0}^{n-1} f_k\, \omega^{-km} \quad \text{where } f_k=f(2k\pi/n)$$

just like interpolation!