CSE 421
Algorithms

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Lecture 13, Autumn 2019

Recurrences, Part 2

Announcements

- Midterm, Oct 30
  - 50 minutes
  - In class
  - Closed book
  - No calculators

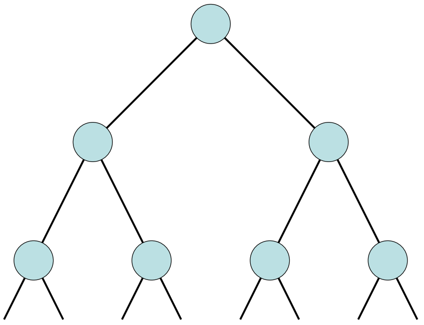
Mergesort Recurrence

- T(n) = 2 T(n/2) + n; T(1) = 1
  - O(n log n)

- Useful facts:
  - $\log_k n = \log_2 n / \log_2 k$
  - $k^{\log n} = n^{\log k}$

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

Unrolling the recurrence



T(n) = aT(n/b) + f(n)

T(n) = T(n/2) + cn

Where does this recurrence arise?

Solving the recurrence exactly

Recursive Matrix Multiplication

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

r = ae + bf
s = ag + bh
t = ce + df
u = cg + dh

A N x N matrix can be viewed as a 2 x 2 matrix with entries that are (N/2) x (N/2) matrices.

The recursive matrix multiplication algorithm recursively multiplies the (N/2) x (N/2) matrices and combines them using the equations for multiplying 2 x 2 matrices

Recursive Matrix Multiplication

- How many recursive calls are made at each level?
- How much work in combining the results?
- What is the recurrence?

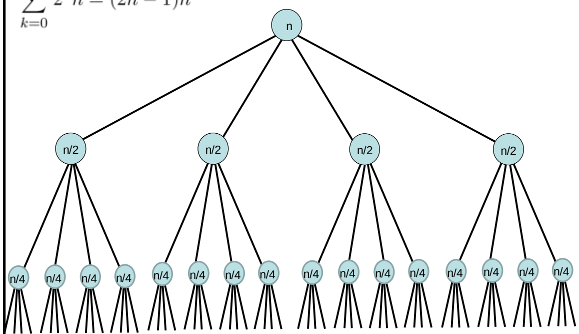
What is the run time for the recursive Matrix Multiplication Algorithm?

- Recurrence:

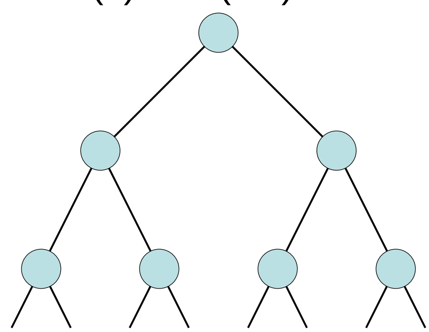
Total Work

$$\sum_{k=0}^{\log n} 2^k n = (2n-1)n$$

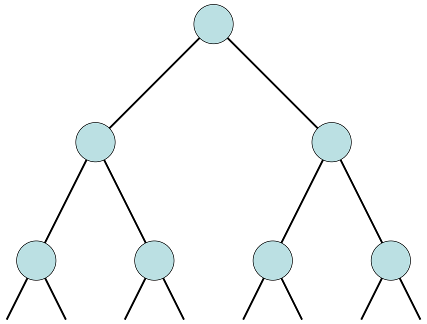
T(n) = 4T(n/2) + n



T(n) = 2T(n/2) + n^2



$T(n) = 2T(n/2) + n^{1/2}$



Recurrences

- Three basic behaviors
  - Dominated by initial case
  - Dominated by base case
  - All cases equal – we care about the depth

What you really need to know about recurrences

- Work per level changes geometrically with the level
- Geometrically increasing ($x > 1$)
  - The bottom level wins
- Geometrically decreasing ($x < 1$)
  - The top level wins
- Balanced ($x = 1$)
  - Equal contribution

Classify the following recurrences (Increasing, Decreasing, Balanced)

- $T(n) = n + 5T(n/8)$
- $T(n) = n + 9T(n/8)$
- $T(n) = n^2 + 4T(n/2)$
- $T(n) = n^3 + 7T(n/2)$
- $T(n) = n^{1/2} + 3T(n/4)$

Strassen's Algorithm

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = p_1 + p_2 - p_4 + p_6$

$s = p_4 + p_5$

$t = p_6 + p_7$

$u = p_2 - p_3 + p_5 - p_7$

Where:

$p_1 = (b - d)(f + h)$

$p_2 = (a + d)(e + h)$

$p_3 = (a - c)(e + g)$

$p_4 = (a + b)h$

$p_5 = a(g - h)$

$p_6 = d(f - e)$

$p_7 = (c + d)e$

From AHU 1974

Recurrence for Strassen's Algorithms

- $T(n) = 7\,T(n/2) + cn^2$
- What is the runtime?

$\log_2 7 = 2.8073549221$

BFPRT Recurrence

$$\boxed{T(n) <= T(3n/4) + T(n/5) + 20\ n}$$

What bound do you expect?