CSE 421: Introduction to Algorithms

Lecture 6

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Generic Graph Traversal Algorithm

Find: set **R** of vertices reachable from $\mathbf{s} \in \mathbf{V}$

Reachable(**s**):

$\mathbf{R} \leftarrow \{\mathbf{s}\}$

While there is a $(\mathbf{u},\mathbf{v}) \in \mathbf{E}$ where $\mathbf{u} \in \mathbf{R}$ and $\mathbf{v} \notin \mathbf{R}$

Add **v** to **R**

Return **R**

Graph Traversal

- Learn the basic structure of a graph
- Walk from a fixed starting vertex **s** to find all vertices reachable from **s**

- Three states of vertices
  - **unvisited**
  - **visited/discovered** (in **R**)
  - **fully-explored** (in **R** and all neighbors in **R**)

Last time: Breadth-First Search

- Completely explore the vertices in order of their distance from **s**

- Naturally implemented using a queue

BFS(s)

Global initialization: mark all vertices “unvisited”

BFS(**s**)

    mark **s** “visited”; $\mathbf{R} \leftarrow \{\mathbf{s}\}$; layer $\mathbf{L_0} \leftarrow \{\mathbf{s}\}$

    while $\mathbf{L_i}$ not empty

        $\mathbf{L_{i+1}} \leftarrow \varnothing$

        For each $\mathbf{u} \in \mathbf{L_i}$

            for each edge $\{\mathbf{u},\mathbf{v}\}$

                if (**v** is “unvisited”)

                    mark **v** “visited”

                    Add **v** to set **R** and to layer $\mathbf{L_{i+1}}$

            mark **u** “fully-explored”

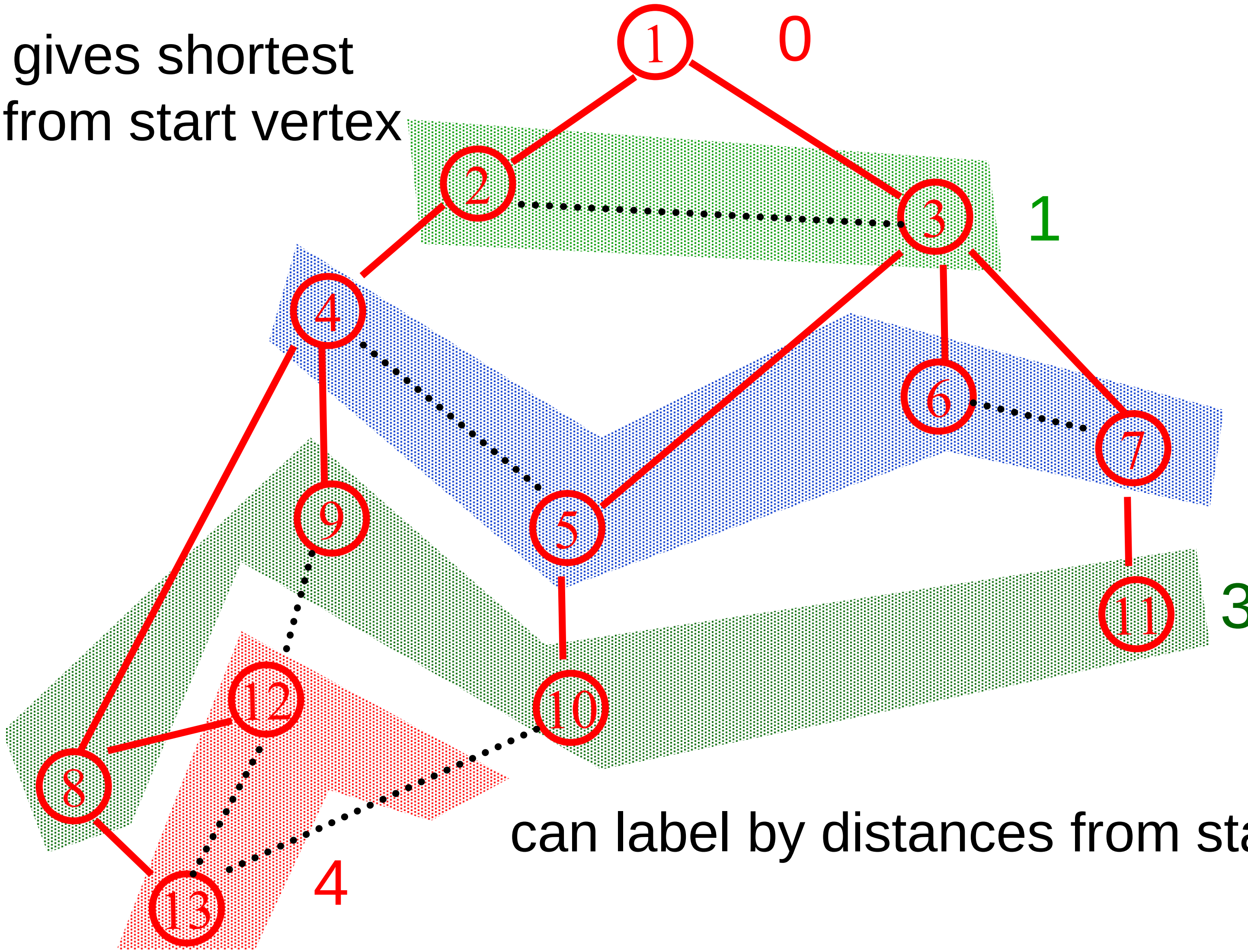
        $\mathbf{i} \leftarrow \mathbf{i+1}$

Properties of BFS

- BFS tree:
  - The rooted tree of edges followed when the BFS algorithm first discovers each vertex
  - Tree for BFS(**s**) gives the shortest paths (in # of hops) from **s** to each vertex reachable from **s**.
- On undirected graphs
  - All non-tree BFS edges join vertices on the same or adjacent layers

BFS Application: Shortest Paths

Tree gives shortest paths from start vertex



can label by distances from start

BFS gives Bipartiteness

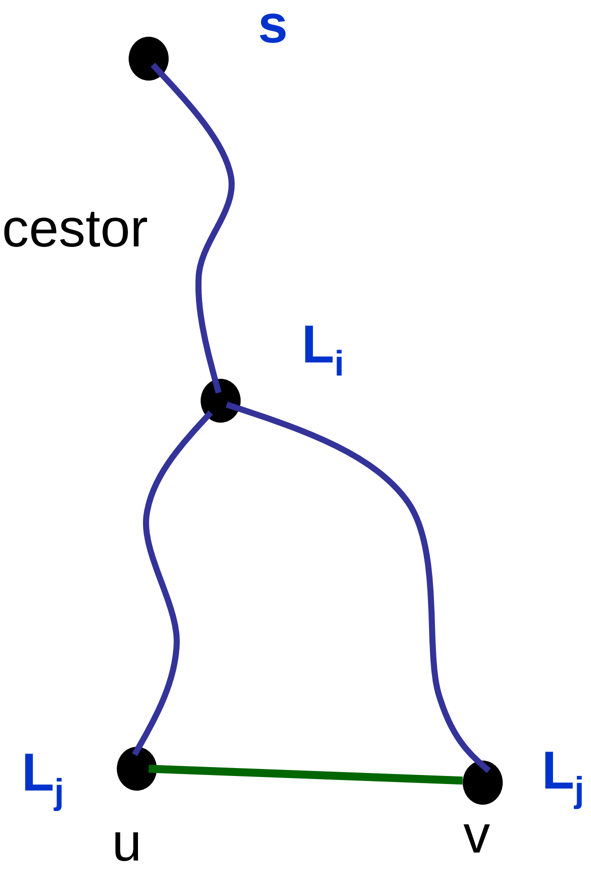
- Run BFS assigning all vertices from layer L_i the color **i** mod **2**
  - i.e. red if they are in an even layer, blue if in an odd layer

- If there is an edge joining two vertices **u** and **v** from the same layer then output "Not Bipartite"

Why does it work?

u and v have a common ancestor

Cycle length **2(j-i)+1**



Graph Search Application: Connected Components

- Want to answer questions of the form:
  - Given: vertices **u** and **v** in **G**
  - Is there a path from **u** to **v**?

- Idea: create array **A** such that

  A[**u**] = smallest numbered vertex that is connected to **u**

  - question reduces to whether **A[u]=A[v]**?

Q: Why not create an array **Path**[**u**,**v**]?

Graph Search Application: Connected Components

- initial state: all **v** unvisited
  for **s**$\leftarrow$**1** to **n** do
      if state(**s**) $\neq$ "fully-explored" then
          BFS(**s**): setting **A**[**u**] $\leftarrow$**s** for each **u** found
              (and marking **u** visited/fully-explored)
      endif
  endfor

- Total cost: **O(n+m)**
  - each vertex is touched once in this outer procedure and the edges examined in the different BFS runs are disjoint
  - works also with "Depth First Search"

DFS(u) – Recursive version

Global Initialization: mark all vertices "unvisited"

DFS(**u**)

    mark **u** “visited” and add **u** to **R**

    for each edge {**u**,**v**}

        if (**v** is “unvisited”)

            DFS(**v**)

    end for

    mark **u** “fully-explored”

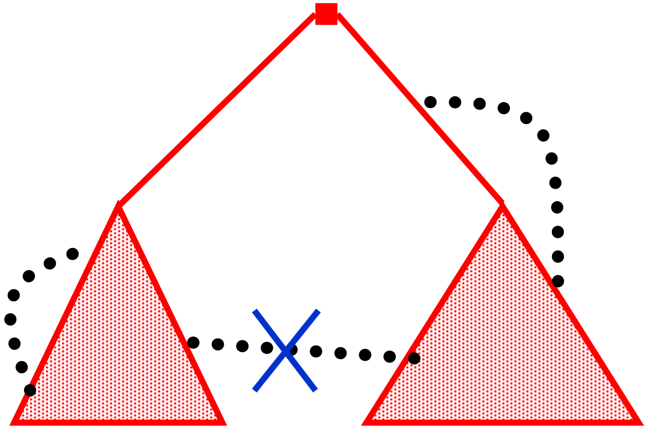
Properties of DFS(s)

- Like BFS(**s**):
  - DFS(**s**) visits **x** if and only if there is a path in **G** from **s** to **x**
  - Edges into undiscovered vertices define a "depth first spanning tree" of **G**
- Unlike the BFS tree:
  - the DFS spanning tree isn't minimum depth
  - its levels don't reflect min distance from the root
  - non-tree edges never join vertices on the same or adjacent levels
- BUT…

Non-tree edges

- All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

- No cross edges.

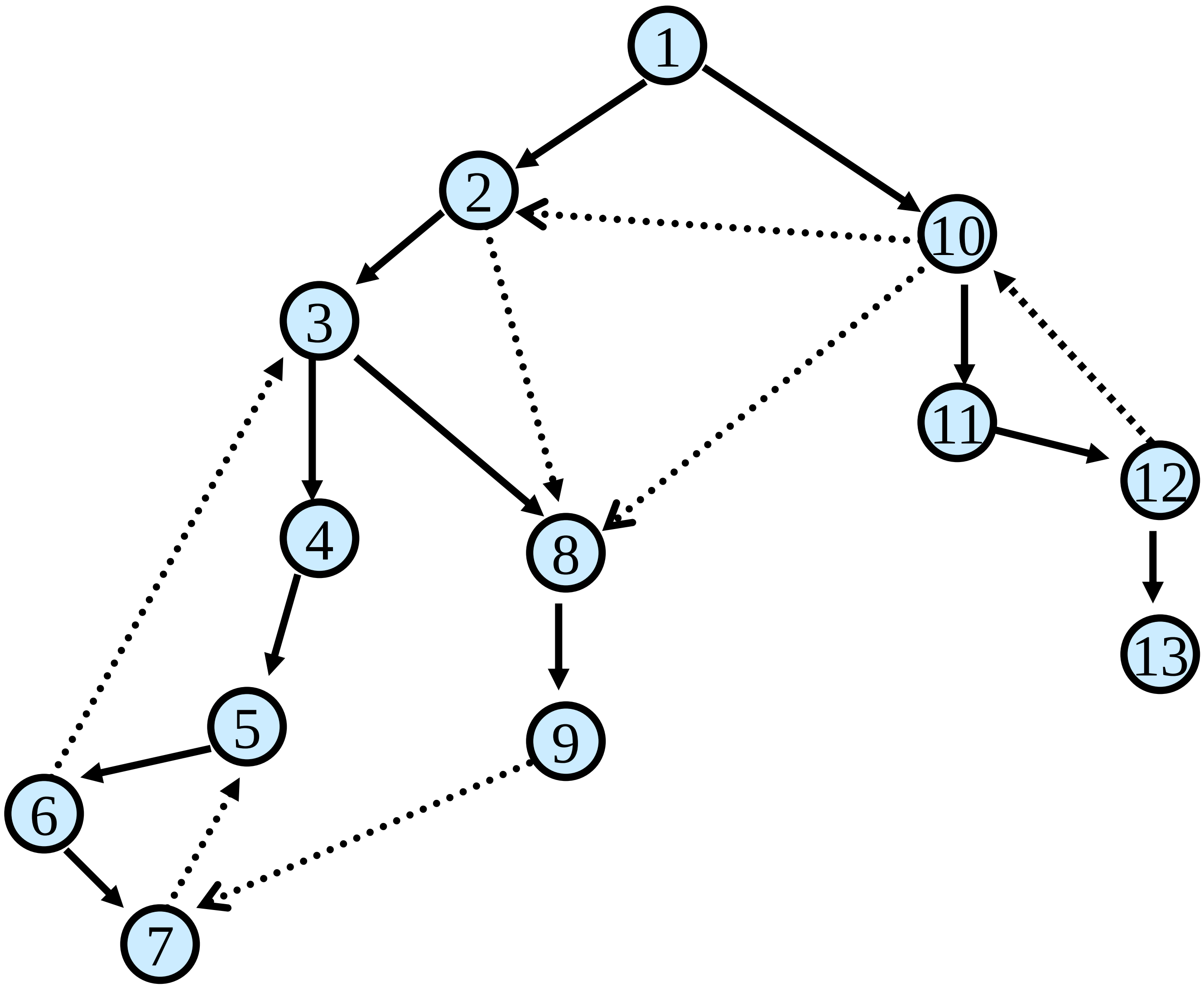


No cross edges in DFS on undirected graphs

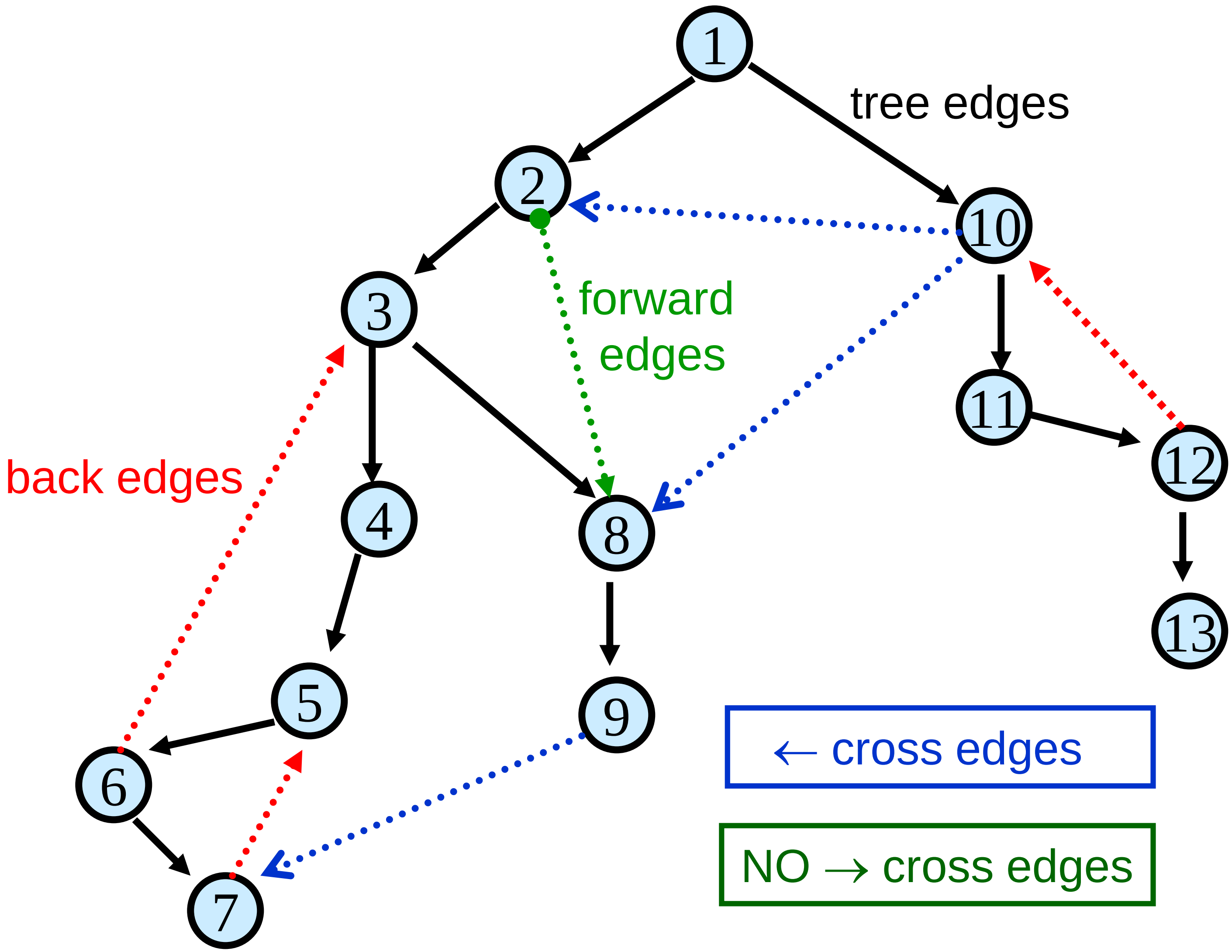
- Claim: During **DFS(x)** every vertex marked visited is a descendant of **x** in the DFS tree **T**

- Claim: For every **x,y** in the DFS tree **T**, if **(x,y)** is an edge not in **T** then one of **x** or **y** is an ancestor of the other in **T**
- Proof:
  - One of **x** or **y** is visited first, suppose WLOG that **x** is visited first and therefore **DFS(x)** was called before **DFS(y)**
    - During **DFS(x)**, the edge **(x,y)** is examined
  - Since **(x,y)** is a not an edge of **T**, **y** was visited when the edge **(x,y)** was examined during **DFS(x)**
  - Therefore **y** was visited during the call to **DFS(x)** so **y** is a descendant of **x**.

DFS(v) for a directed graph



DFS(v)



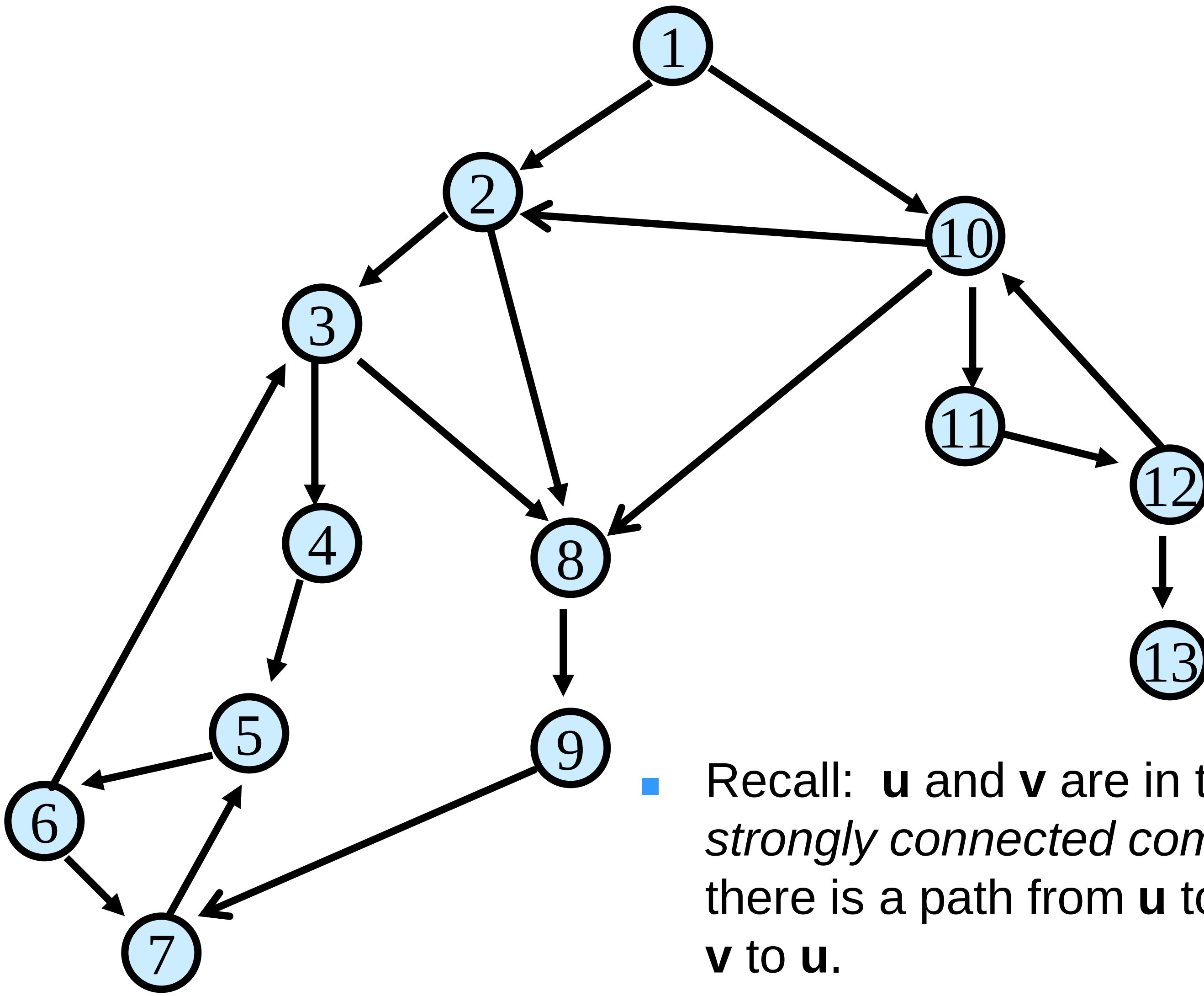
← cross edges

NO → cross edges

Properties of Directed DFS

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s
- Every cycle contains a back edge in the DFS tree

Identify the Strongly Connected Components

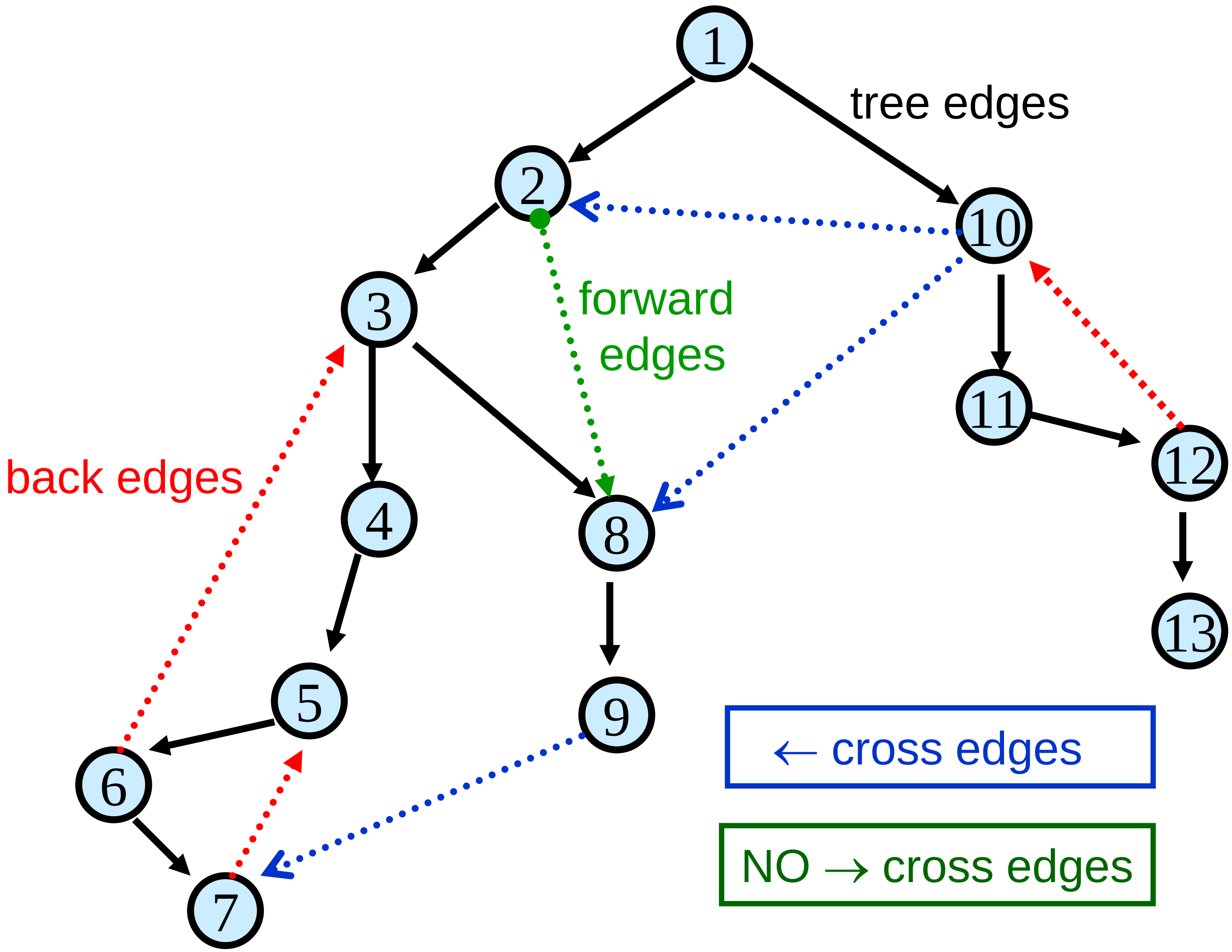


- Recall: **u** and **v** are in the same *strongly connected component* iff there is a path from **u** to **v** and from **v** to **u**.

Strongly connected components can be found in O(n+m) time

- But it's tricky!
- Simpler problem: given a vertex **v**, compute the vertices in **v**'s scc in **O**(**n**+**m**) time

Strongly Connected Components



Directed Acyclic Graphs

- A directed graph $G=(V,E)$ is acyclic if it has no directed cycles

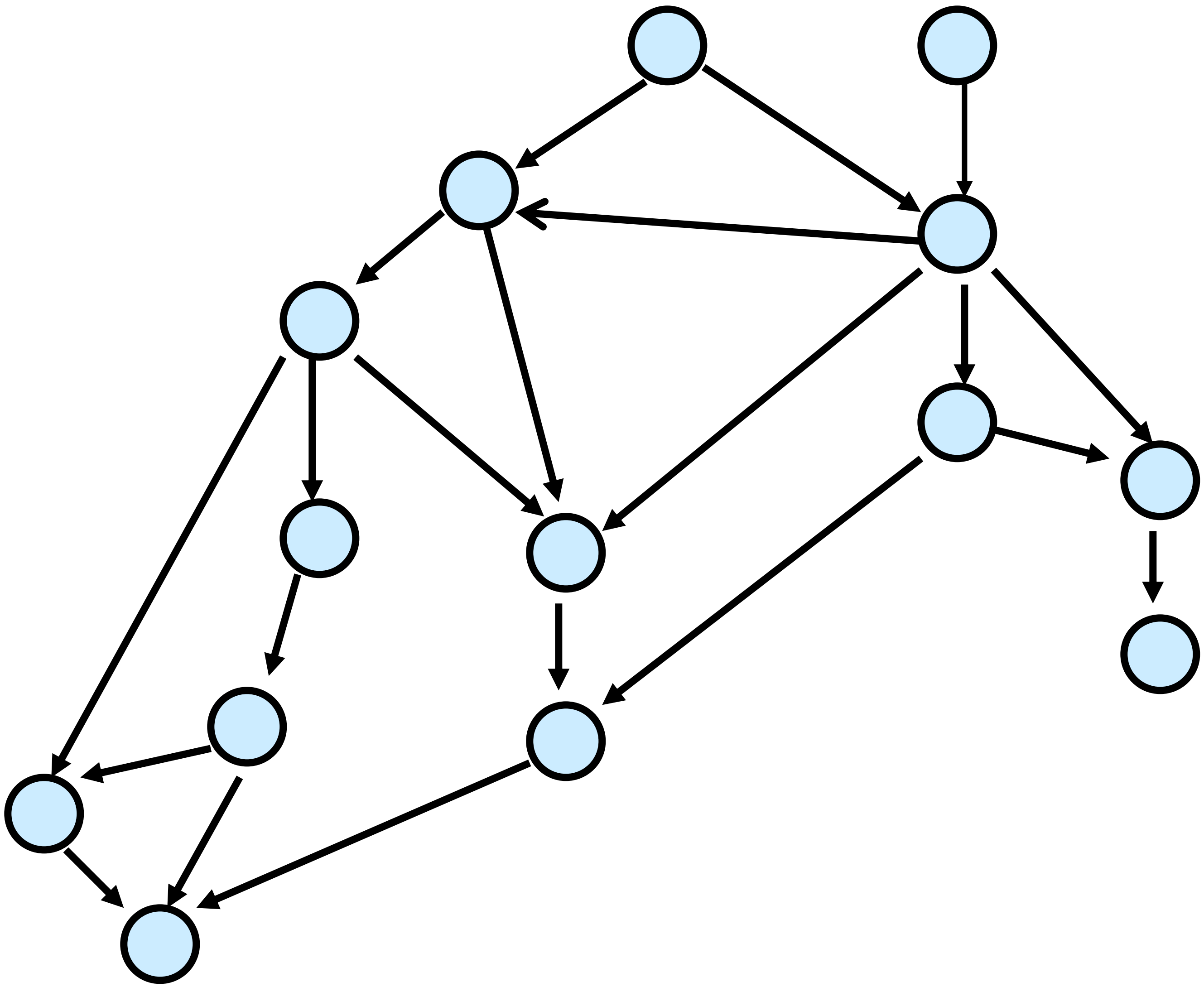
- Terminology: A directed acyclic graph is also called a DAG

Topological Sort

- **Given:** a directed acyclic graph (DAG) **G=(V,E)**
- **Output:** numbering of the vertices of **G** with distinct numbers from **1** to **n** so edges only go from lower number to higher numbered vertices

- Applications
  - nodes represent tasks
  - edges represent precedence between tasks
  - topological sort gives a sequential schedule for solving them

Directed Acyclic Graph



In-degree 0 vertices

- Every DAG has a vertex of in-degree 0
- Proof: By contradiction
  - Suppose every vertex has some incoming edge
  - Consider following procedure:

    while (true) do

        v$\leftarrow$some predecessor of **v**

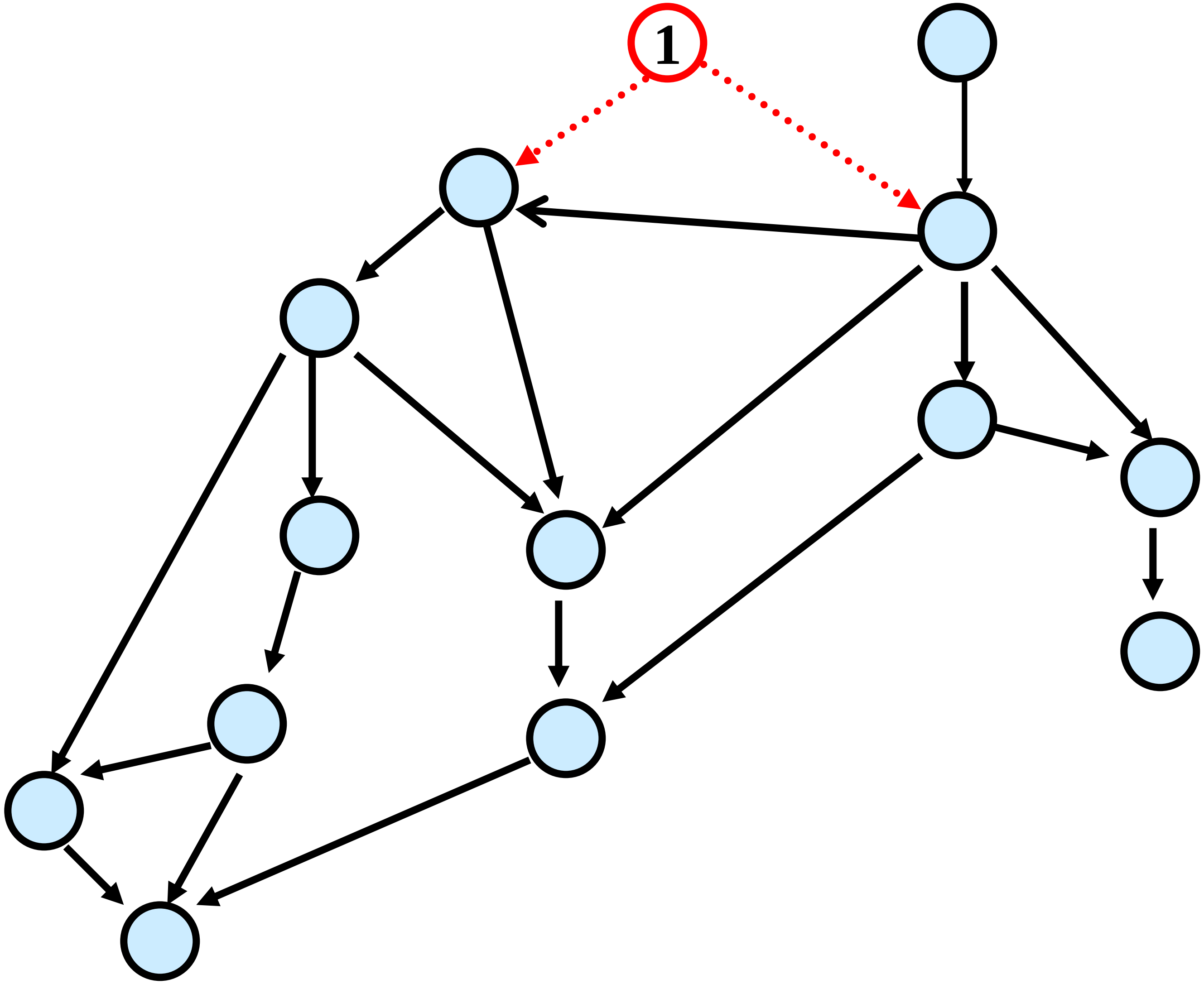
  - After **n+1** steps where **n**=**|V|** there will be a repeated vertex
    - This yields a cycle, contradicting that it is a DAG

Topological Sort

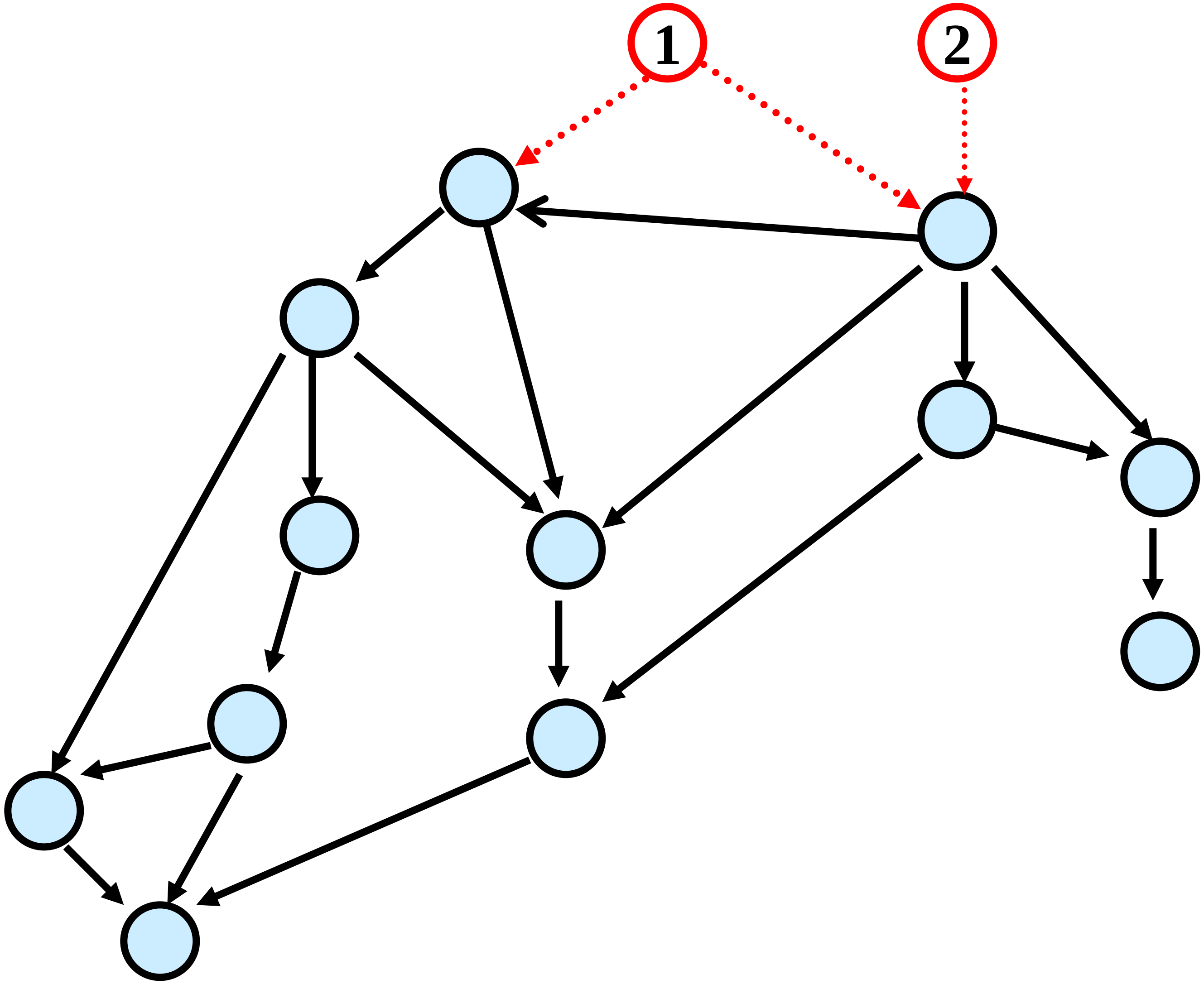
- Can do using DFS

- Alternative simpler idea:
  - Any vertex of in-degree 0 can be given number 1 to start
  - Remove it from the graph and then give a vertex of in-degree 0 number 2, etc.

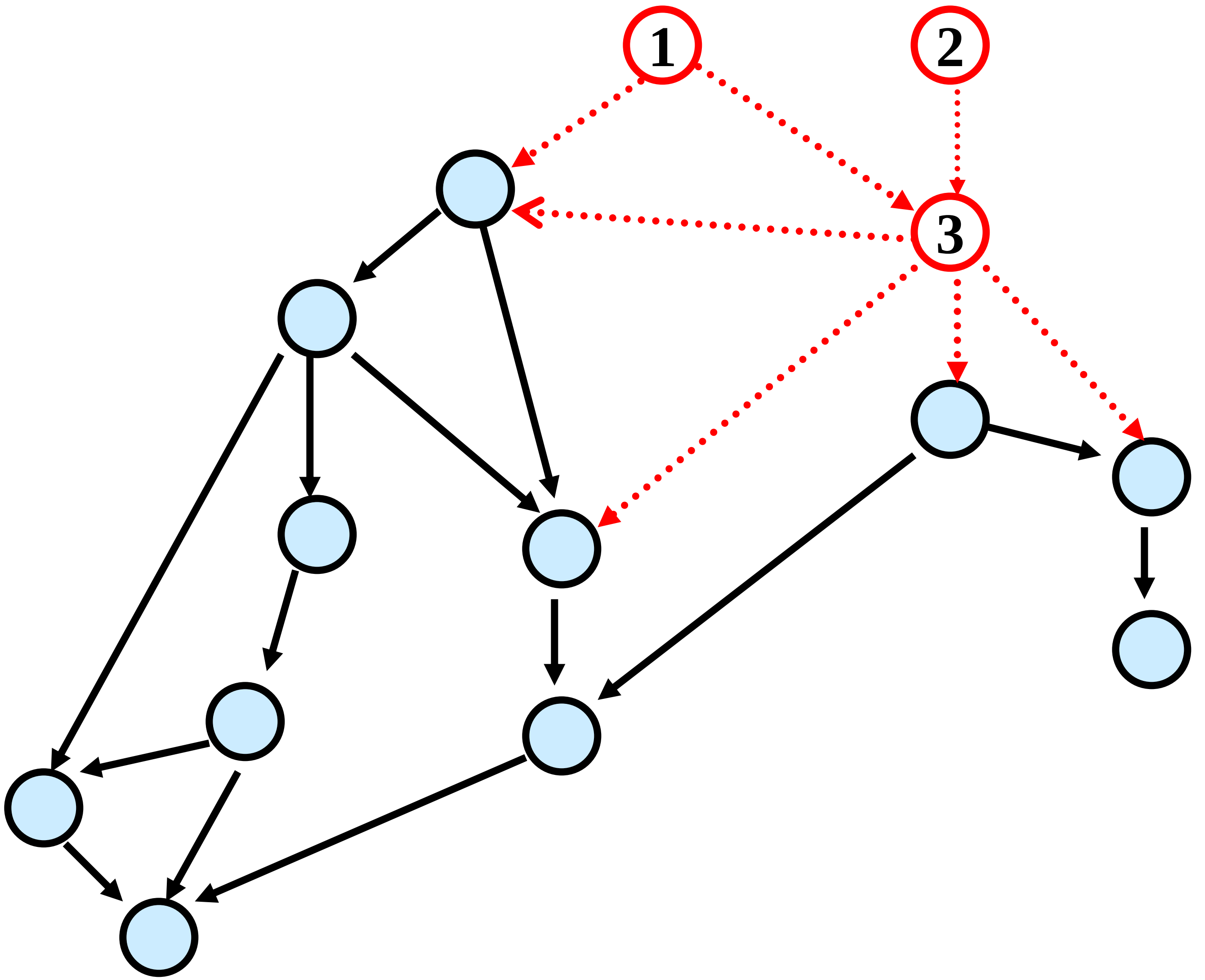
Topological Sort



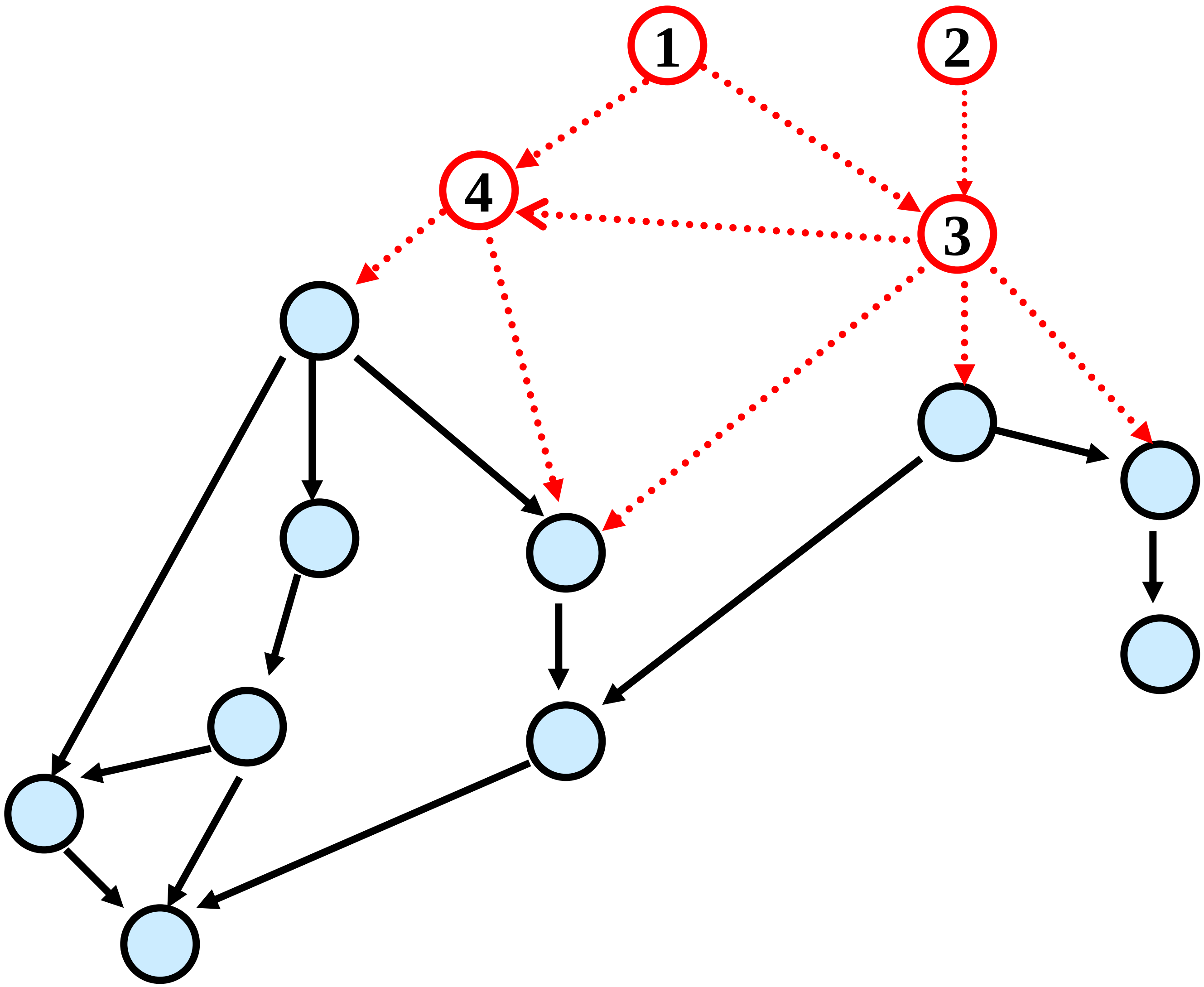
Topological Sort



Topological Sort



Topological Sort



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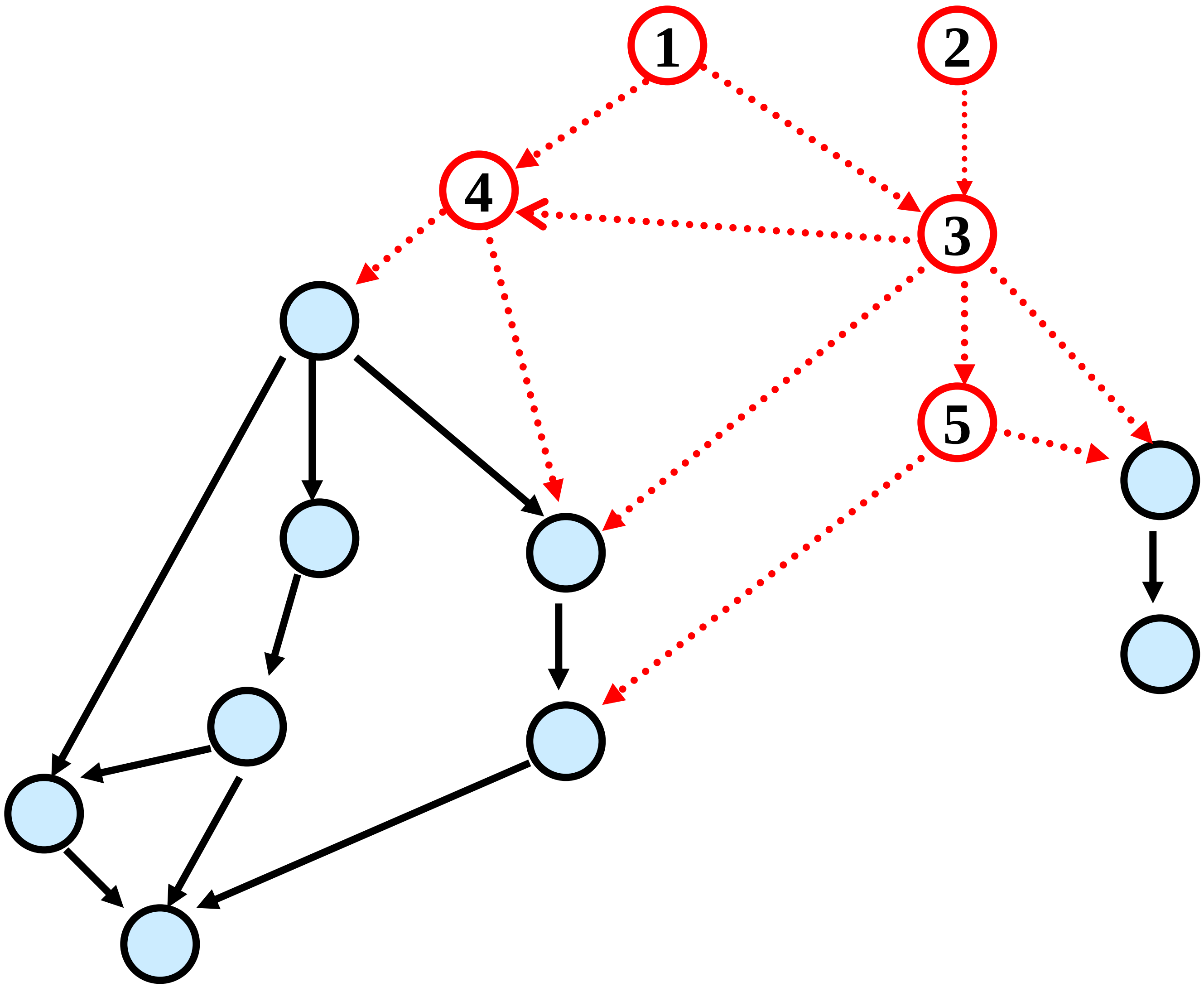
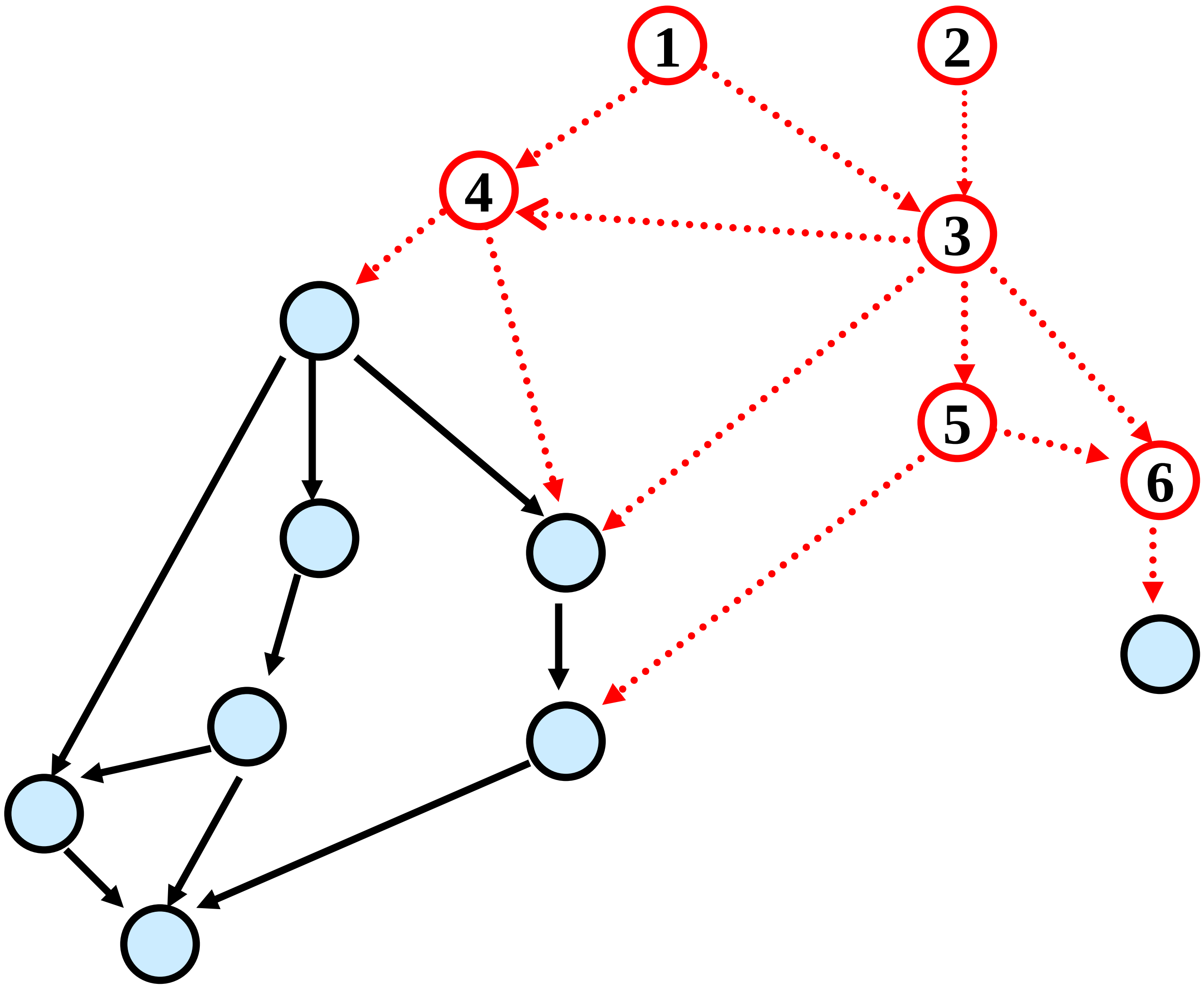


Topological Sort



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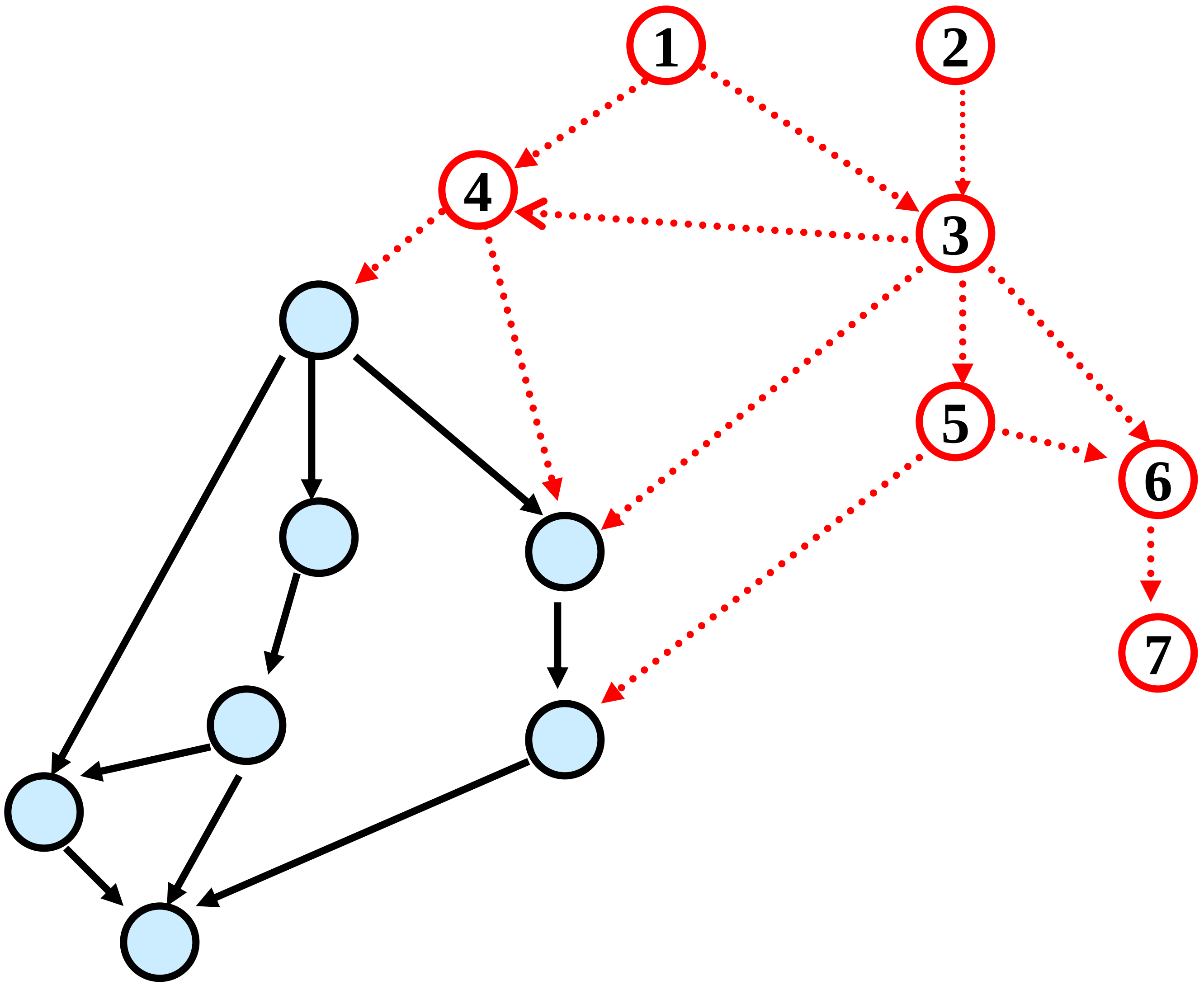

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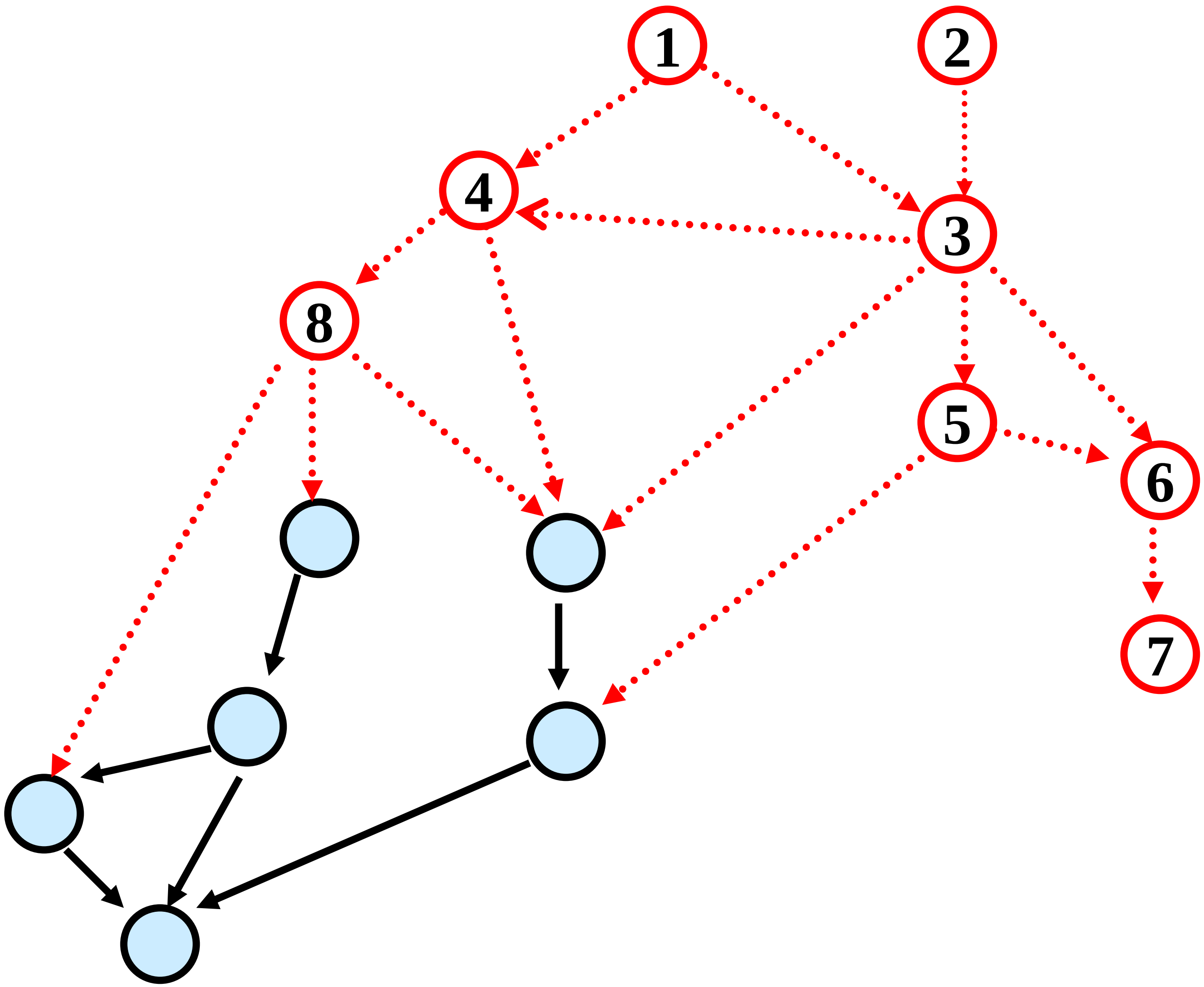
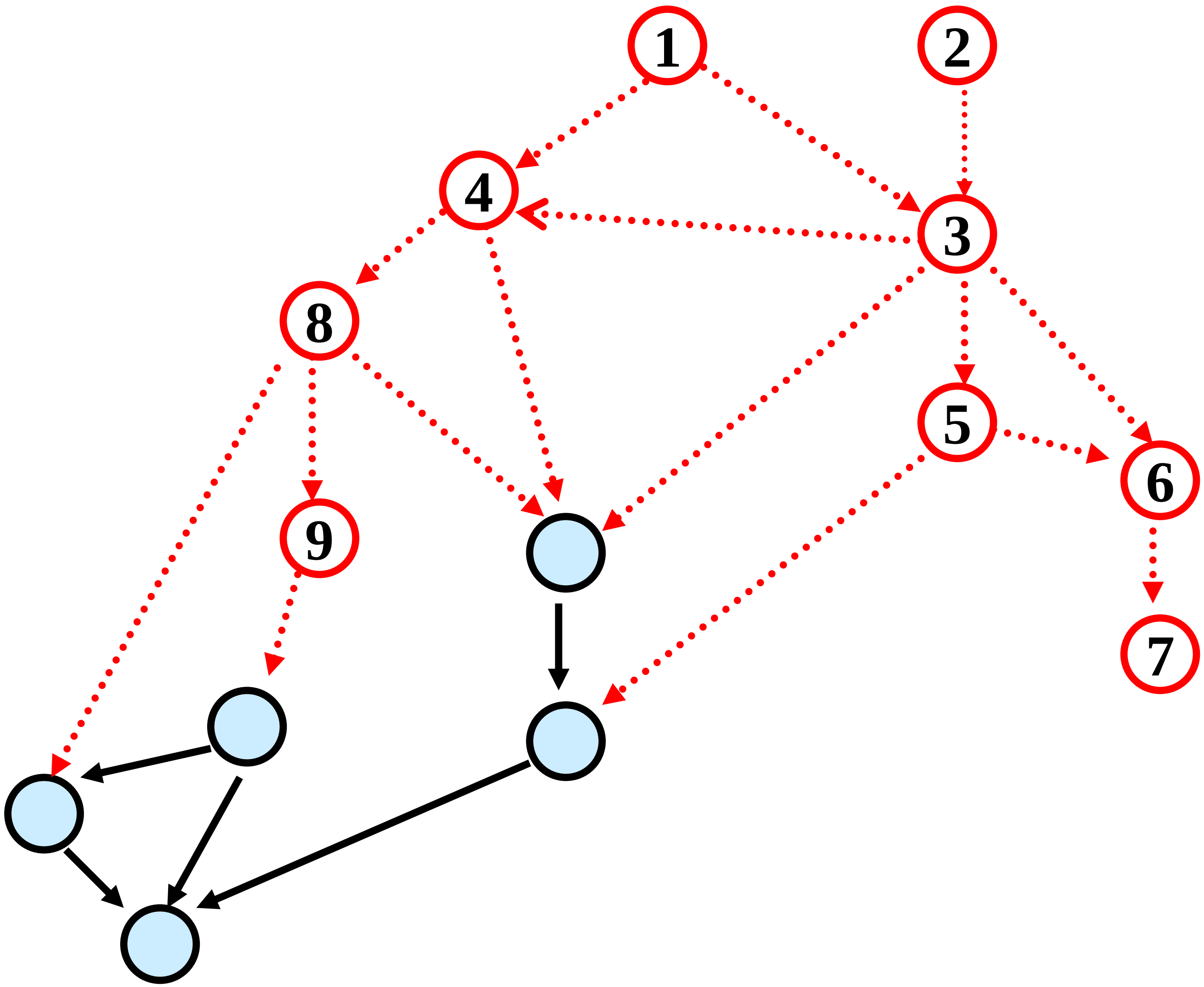


Topological Sort



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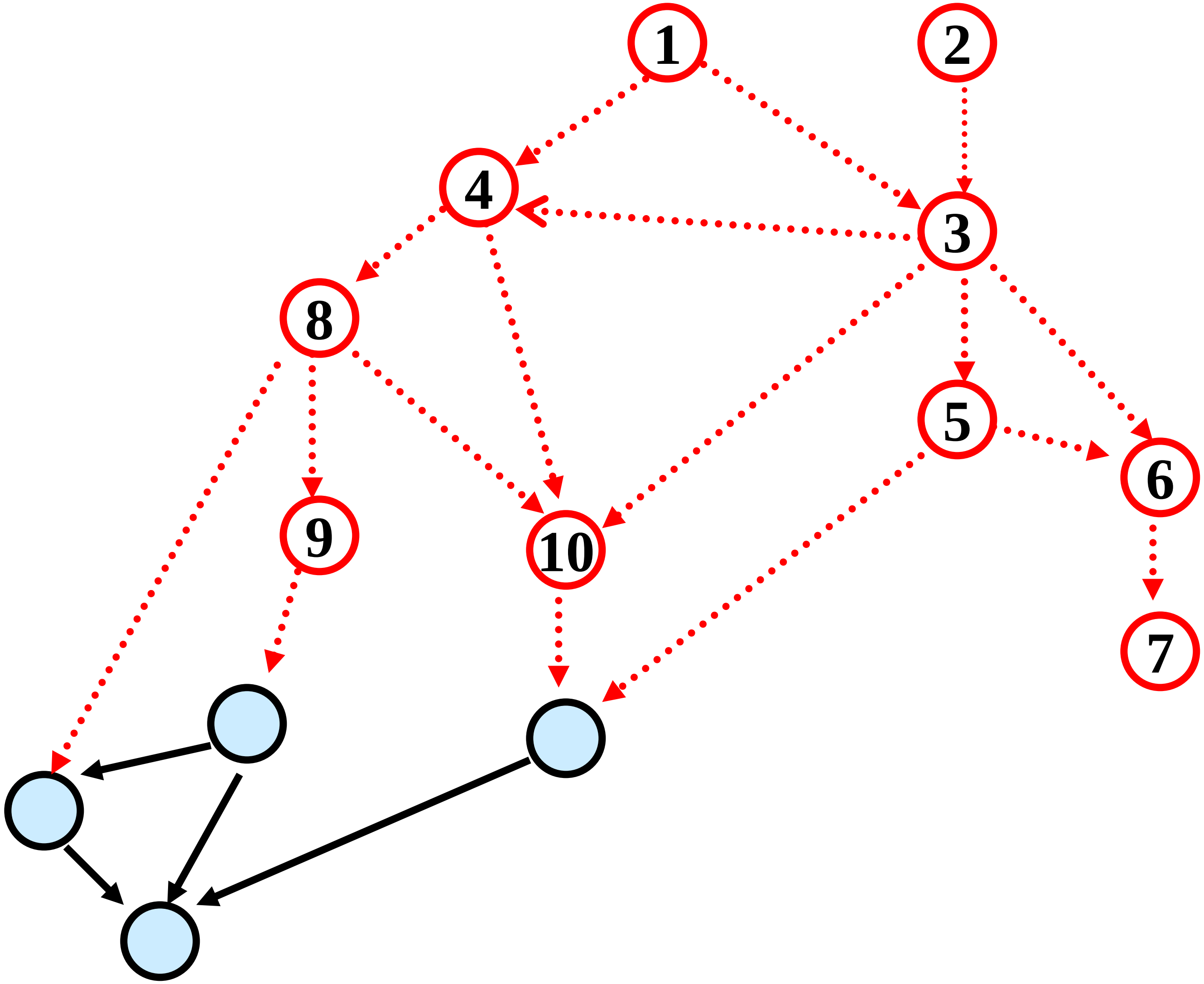

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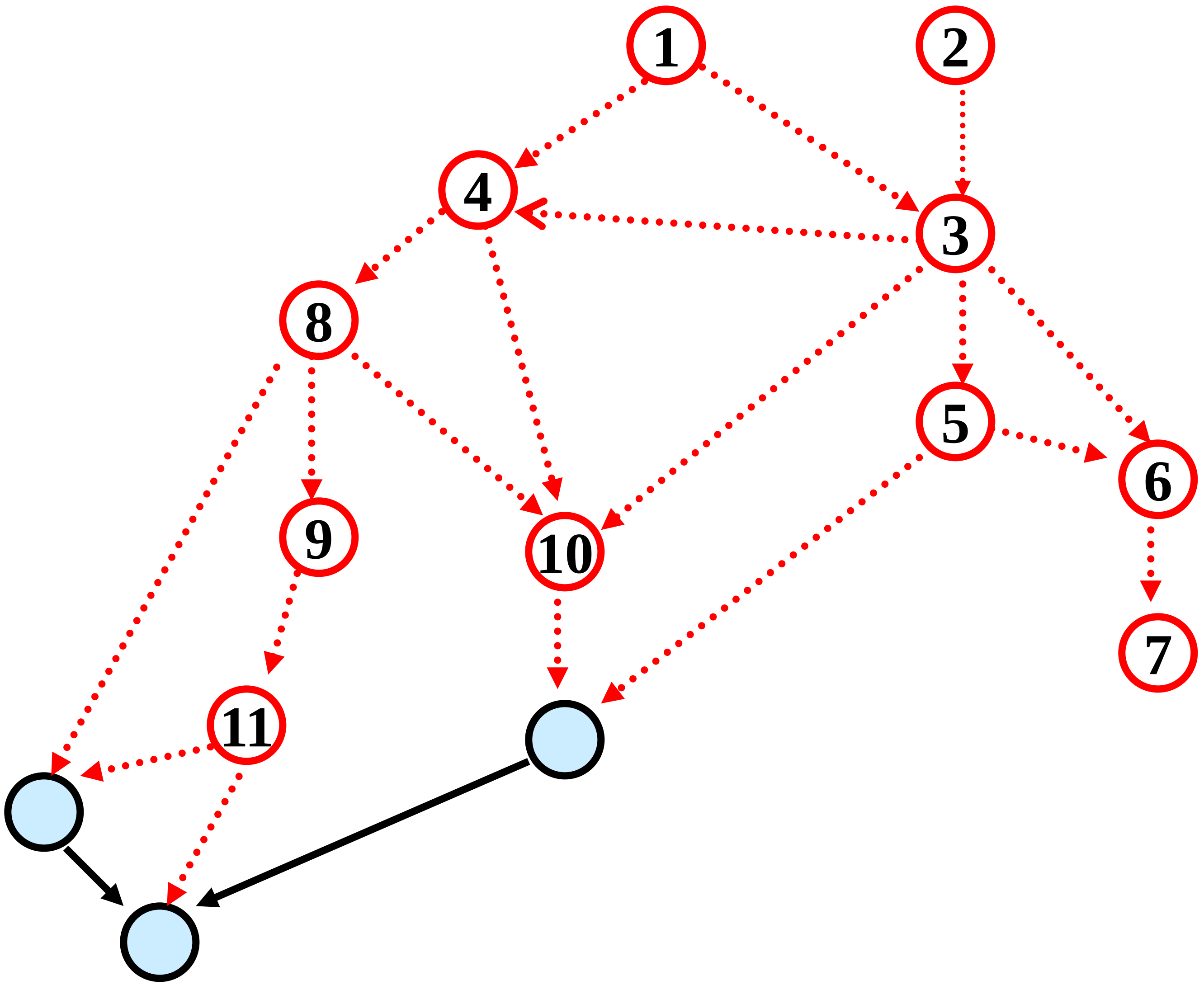

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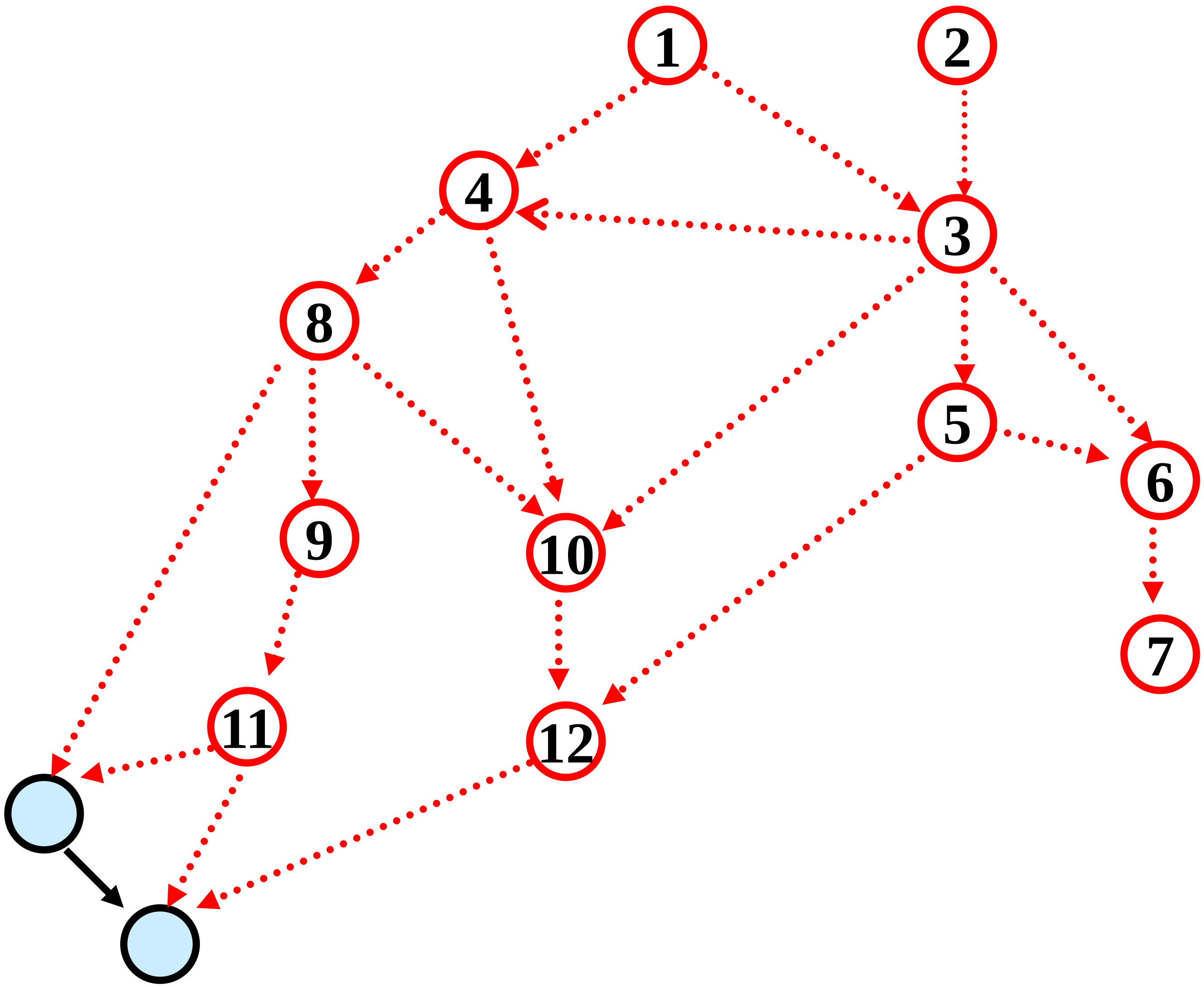

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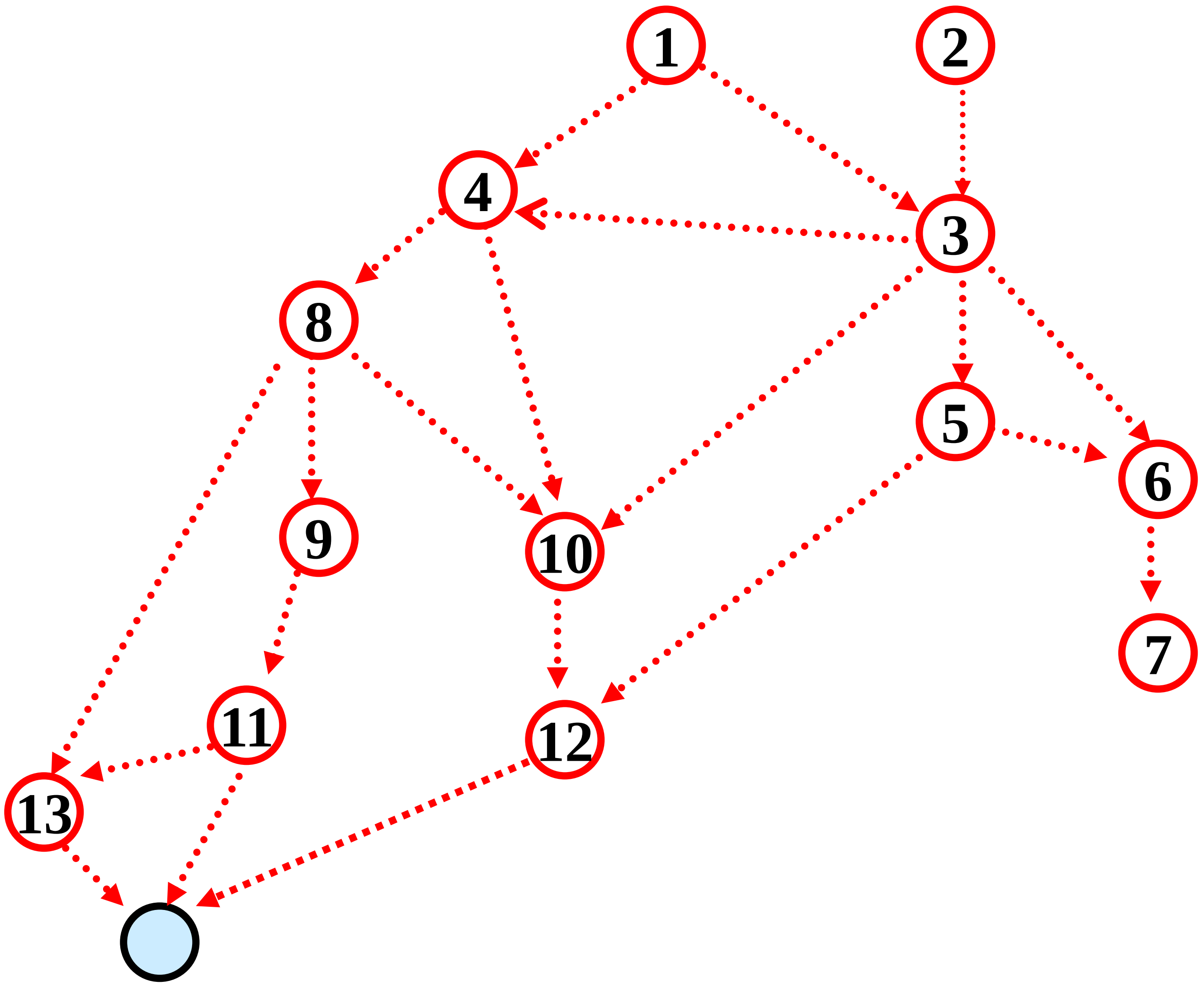

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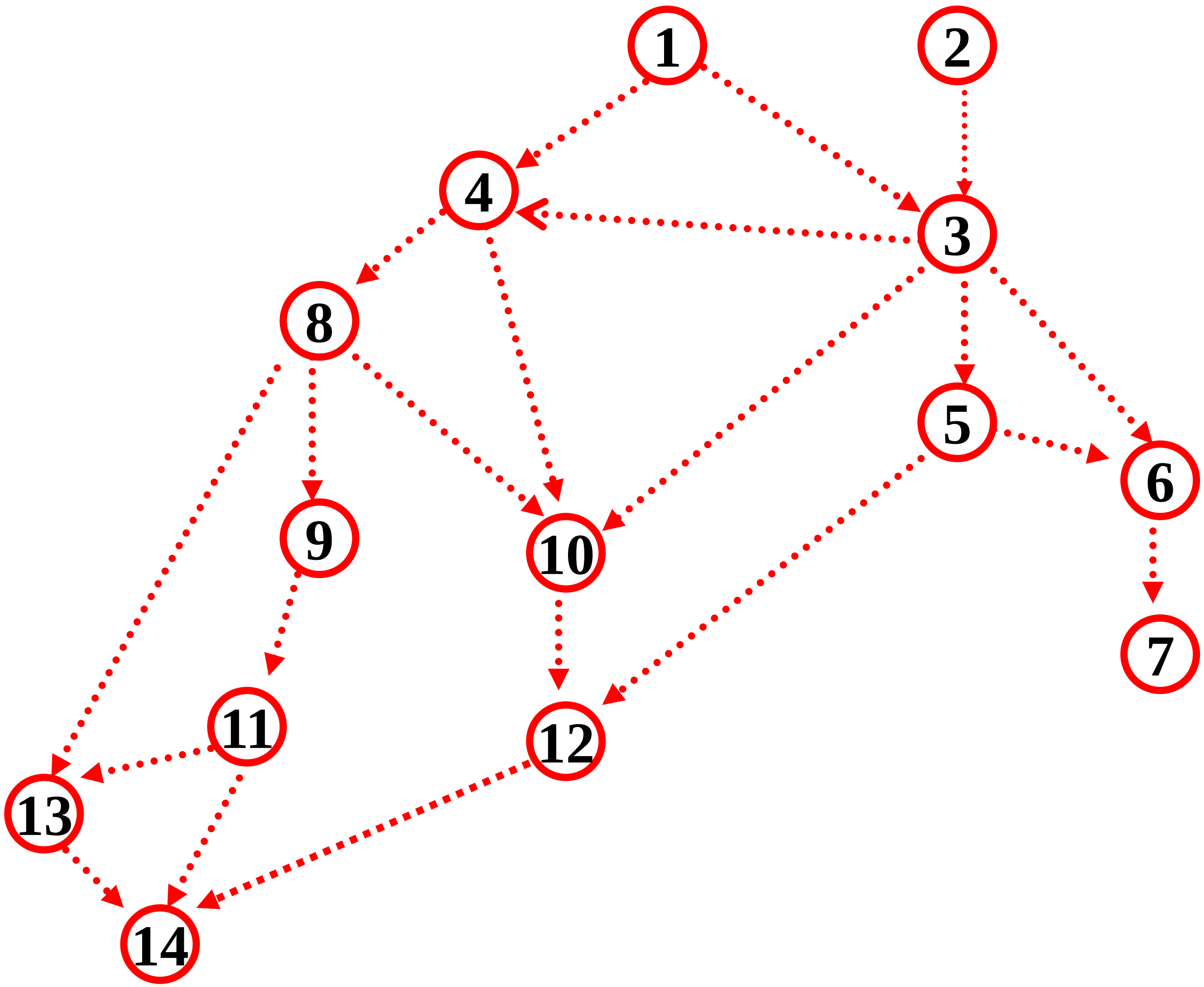

Implementing Topological Sort

- Go through all edges, computing array with in-degree for each vertex **O(m+n)**
- Maintain a queue (or stack) of vertices of in-degree **0**
- Remove any vertex in queue and number it
- When a vertex is removed, decrease in-degree of each of its neighbors by **1** and add them to the queue if their degree drops to **0**

Total cost **O(m+n)**

Strongly Connected Components &Topological Sort work well together

- Shrinking each s.c.c. of directed graph to a vertex yields a DAG
- Can apply topological sort to the resulting DAG to get a good order to deal with the s.c.c.'s of the original graph one at a time