CSE 421

Topological Ordering / Greedy

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DAGs and Topological Ordering
Directed Acyclic Graphs (DAG)

A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

Def: A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge $(v_i, v_j)$ we have $i < j$. 

![Diagram of a DAG and a topological ordering](image)
DAGs: A Sufficient Condition

**Lemma**: If $G$ has a topological order, then $G$ is a DAG.

**Pf.** (by contradiction)

Suppose that $G$ has a topological order $1, 2, \ldots, n$ and that $G$ also has a directed cycle $C$.

Let $i$ be the lowest-indexed node in $C$, and let $j$ be the node just before $i$; thus $(j, i)$ is an (directed) edge. By our choice of $i$, we have $i < j$.

On the other hand, since $(j, i)$ is an edge and $1, 2, \ldots, n$ is a topological order, we must have $j < i$, a contradiction.

Have you seen this idea before?
Yes! In analyzing Man-optimal stable matching
DAGs: A Sufficient Condition

G has a topological order \Rightarrow \text{?} \Rightarrow \text{G is a DAG}
Every DAG has a source node

**Lemma**: If G is a DAG, then G has a node with no incoming edges (i.e., a source).

**Pf.** (by contradiction)
Suppose that G is a DAG and it has no source
Pick any node v, and begin following edges **backward** from v. Since v has at least one incoming edge (u, v) we can walk backward to u. Then, since u has at least one incoming edge (x, u), we can walk backward to x.
Repeat until we visit a node, say w, twice.
Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.

Is this similar to a previous proof?
DAG => Topological Order

Lemma: If G is a DAG, then G has a topological order

Pf. (by induction on n)
Base case: true if n = 1.

IH: Every DAG with n-1 vertices has a topological ordering.

IS: Given DAG with \( n > 1 \) nodes, find a source node v.
\( G - \{ v \} \) is a DAG, since deleting v cannot create cycles.

By IH, \( G - \{ v \} \) has a topological ordering.
Place v first in topological ordering; then append nodes of G - \{ v \}
in topological order. This is valid since v has no incoming edges.

Reminder: Always remove vertices/edges to use IH
A Characterization of DAGs

G has a topological order \iff G is a DAG
Topological Order Algorithm: Example
Topological Order Algorithm: Example

Topological order: 1, 2, 3, 4, 5, 6, 7
Topological Sorting Algorithm

Maintain the following:
- \( \text{count}[w] \) = (remaining) number of incoming edges to node \( w \)
- \( S \) = set of (remaining) nodes with no incoming edges

Initialization:
- \( \text{count}[w] = 0 \) for all \( w \)
- \( \text{count}[w]++ \) for all edges \((v,w)\)
- \( S = S \cup \{w\} \) for all \( w \) with \( \text{count}[w]=0 \)

Main loop:
- while \( S \) not empty
  - remove some \( v \) from \( S \)
  - make \( v \) next in topo order
  - for all edges from \( v \) to some \( w \)
    - decrement \( \text{count}[w] \)
    - add \( w \) to \( S \) if \( \text{count}[w] \) hits 0

Correctness: clear, I hope

Time: \( O(m + n) \) (assuming edge-list representation of graph)
DFS on Directed Graphs

• Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s

• Every cycle contains a back edge in the DFS tree
Summary

- Graphs: abstract relationships among pairs of objects
- Terminology: node/vertex/vertices, edges, paths, multi-edges, self-loops, connected
- Representation: Adjacency list, adjacency matrix
- Nodes vs Edges: \( m = O(n^2) \), often less
- BFS: Layers, queue, shortest paths, all edges go to same or adjacent layer
- DFS: recursion/stack; all edges ancestor/descendant
- Algorithms: Connected Comp, bipartiteness, topological sort
Greedy Algorithms
Greedy Strategy

**Goal:** Given currency denominations: 1, 5, 10, 25, 100, give change to customer using *fewest* number of coins.

**Ex:** 34¢.

**Cashier's algorithm:** At each iteration, give the *largest* coin valued ≤ the amount to be paid.

**Ex:** $2.89.
Greedy is not always Optimal

Observation: Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.
   Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
   Optimal: 70, 70.

Lesson: Greedy is short-sighted. Always chooses the most attractive choice at the moment. But this may lead to a dead-end later.
Greedy Algorithms Outline

Pros
• Intuitive
• Often simple to design (and to implement)
• Often fast

Cons
• Often incorrect!

Proof techniques:
• Stay ahead
• Structural
• Exchange arguments
Interval Scheduling
Interval Scheduling

- Job $j$ starts at $s(j)$ and finishes at $f(j)$.
- Two jobs compatible if they don’t overlap.
- Goal: find maximum subset of mutually compatible jobs.
Greedy Strategy

Sort the jobs in some order. Go over the jobs and take as much as possible provided it is compatible with the jobs already taken.

Main question:

• What order?

• Does it give the Optimum answer?

• Why?
Possible Approaches for Inter Sched

Sort the jobs in some order. Go over the jobs and take as much as possible provided it is compatible with the jobs already taken.

[Earliest start time] Consider jobs in ascending order of start time $s_j$.

[Earliest finish time] Consider jobs in ascending order of finish time $f_j$.

[Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.

[Fewest conflicts] For each job, count the number of conflicting jobs $c_j$. Schedule in ascending order of conflicts $c_j$. 
Greedy Alg: Earliest Finish Time

Consider jobs in increasing order of finish time. Take each job provided it’s compatible with the ones already taken.

Sort jobs by finish times so that \( f(1) \leq f(2) \leq \ldots \leq f(n) \).

\( A \) \( \emptyset \)

for \( j = 1 \) to \( n \) {
    if (job \( j \) compatible with \( A \))
        \( A \leftarrow A \cup \{j\} \)
}

return \( A \)

Implementation. \( O(n \log n) \).

- Remember job \( j^* \) that was added last to \( A \).
- Job \( j \) is compatible with \( A \) if \( s(j) \geq f(j^*) \).
Greedy Alg: Example
Correctness

Theorem: Greedy algorithm is optimal.

Pf: (technique: “Greedy stays ahead”)
Let \(i_1, i_2, \ldots, i_k\) be jobs picked by greedy, \(j_1, j_2, \ldots, j_m\) those in some optimal solution in order. We show \(f(i_r) \leq f(j_r)\) for all \(r\), by induction on \(r\).

Base Case: \(i_1\) chosen to have min finish time, so \(f(i_1) \leq f(j_1)\).

IH: \(f(i_r) \leq f(j_r)\) for some \(r\)

IS: Since \(f(i_r) \leq f(j_r) \leq s(j_r + 1)\), \(j_{r+1}\) is among the candidates considered by greedy when it picked \(i_{r+1}\), & it picks min finish, so \(f(i_{r+1}) \leq f(j_{r+1})\)

Observe that we must have \(k \geq m\), else \(j_{k+1}\) is among (nonempty) set of candidates for \(i_{k+1}\)