Claim: Suppose \( y \) is discovered during \( \text{DFS}(x) \). Then \( y \) is a descendant of \( x \).

Lemma: Say \( T \) is \( \text{DFS}(S) \). Suppose \( \{x,y\} \in T \). (\( \{x,y\} \in E \)).

Then \( x \) is an ancestor of \( y \) or vice versa.

Pf. W.L.O.G. assume \( x \) is discovered first.

We call \( \text{DFS}(x) \). (at this point \( y \) is undiscovered).

By prev. claim enough to show \( y \) is discovered during \( \text{DFS}(x) \).

By the time the For loop of \( \text{DFS}(x) \) gets \( y \), \( y \) must be discovered (o.w. edge \( \{x,y\} \in T \) and it is a contradiction).

So \( y \) is discovered during \( \text{DFS}(x) \).

For all plan graphs

\[ m \leq 3n - 4 \]

Hint: Any plan has a vertex \( v \) \( \deg(v) \leq 5 \).

Then, use induction.

Claim: Any plan graph has a vertex \( v \): \( \deg(v) \leq 5 \).

\[ \sum \deg(v) = 2m \leq 6n - 8 \]

\[ \Rightarrow \exists v: \deg(v) \leq 6 \]

II. A plan graph \( G \) with \( n - 1 \) vertices, color with \( 6 \) colors.

IS. Give plan graph \( G \) with \( n \) vertices.
\( f \) of degree \( \leq 5 \) remove \( v \). \( G - \{ v \} \) is planar because you can draw on plan.

So by IH we can color \( G - \{ v \} \).

We color \( v \) with a color not appear in its neighbors.

We can do that by \( d(y(v)) \leq 5 \).

**Claim:** If \( G \) has a topological order then it is a DAG.

**pf.**

\[
\begin{align*}
1 & \rightarrow 2 \quad \Downarrow \\
& \quad \downarrow \\
& \quad v
\end{align*}
\]