Lecture 8

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1 In-class Exercise: Coloring Planar graphs

Theorem 1. *The vertices of any planar graph can be colored with* 6 *colors in such a way that every edge gets exactly two distinct colors.*

Use the following fact: For any planar graph we have $3n - 4 \geq m$, where n is the number of vertices and m is the number of edges.

First, we prove the following fact:

Claim 2. *In any planar graph there exists a vertex* v *with* $\deg(v) \leq 5$.

Proof First, recall that for any graph G

$$\sum_v \deg(v) = 2m.$$

But since by claim assumption, $2m \leq 6n - 8$, we have $\sum_v \deg(v) \leq 6n - 8$.

We prove by contradiction that there exists a vertex v with $\deg(v) \leq 5$. If for all v, $\deg(v) \geq 6$, then

$$6n - 4 \geq \sum_v \deg(v) \geq 6n$$

which is a contradiction. ■

Now, we prove the theorem by induction:

Base Case: A planar graph with 1 vertex can be colored with 6 colors obviously.

IH: Every planar graph with $n - 1$ vertices can be colored with 6 colors.

IS: We want to show that every planar graph with n vertices can be colored with 6 colors. Let G be a planar graph with n vertices. We show that G can be colored with 6 colors. By claim G has a vertex v with $\deg(v) \leq 5$. Let $H = G - \{v\}$.

We claim that H is also planar. Because if we can draw G on the plane with no crossing, when we remove v and its edges, we still have a drawing of the remaining graph (i.e., H) with no crossing. Therefore, H is a planar graph with $n - 1$ vertices. So, by IH, H can be colored with 6 colors.

Now, let's add vertex v (and its edges) back in. We need to find a consistent color vertex v and this would complete the proof. By definition, v has at most 5 neighbors. Since we have 6 colors, there exists a color which is not used in any of the neighbors of v. We color v with that color and we obtain a consistent coloring.