

1. **In-class Exercise: Coloring Planar graphs**

**Theorem 1.** The vertices of any planar graph can be colored with 6 colors in such a way that every edge gets exactly two distinct colors.

Use the following fact: For any planar graph we have $3n - 4 \geq m$, where $n$ is the number of vertices and $m$ is the number of edges.

First, we prove the following fact:

**Claim 2.** *In any planar graph there exists a vertex $v$ with $\deg(v) \leq 5$.*

**Proof** First, recall that for any graph $G$

$$\sum_v \deg(v) = 2m.$$  

But since by claim assumption, $2m \leq 6n - 8$, we have $\sum_v \deg(v) \leq 6n - 8$.

We prove by contradiction that there exists a vertex $v$ with $\deg(v) \leq 5$. If for all $v$, $\deg(v) \geq 6$, then

$$6n - 4 \geq \sum_v \deg(v) \geq 6n$$

which is a contradiction. $\blacksquare$

Now, we prove the theorem by induction:

**Base Case:** A planar graph with 1 vertex can be colored with 6 colors obviously.

**IH:** Every planar graph with $n - 1$ vertices can be colored with 6 colors.

**IS:** We want to show that every planar graph with $n$ vertices can be colored with 6 colors. Let $G$ be a planar graph with $n$ vertices. We show that $G$ can be colored with 6 colors. By claim $G$ has a vertex $v$ with $\deg(v) \leq 5$. Let $H = G - \{v\}$.

We claim that $H$ is also planar. Because if we can draw $G$ on the plane with no crossing, when we remove $v$ and its edges, we still have a drawing of the remaining graph (i.e., $H$) with no crossing. Therefore, $H$ is a planar graph with $n - 1$ vertices. So, by IH, $H$ can be colored with 6 colors.

Now, let’s add vertex $v$ (and its edges) back in. We need to find a consistent color vertex $v$ and this would complete the proof. By definition, $v$ has at most 5 neighbors. Since we have 6 colors, there exists a color which is not used in any of the neighbors of $v$. We color $v$ with that color and we obtain a consistent coloring.