Lemma: \( L(v) = i \) if and only if shortest path to \( s \) is \( i \).

Proof:

1. \( L(v) = i \) \( \Rightarrow \) shortest path \( \leq i \).

2. Shortest path \( = i \) \( \Rightarrow \) \( L(v) \leq i \).

\( S = v_0, v_1, \ldots, v_i = v \)

\( L(v_0) = 0 \)

\( \forall j \), \( L(v_j) - L(v_{j-1}) \leq 1 \)

\( \frac{1}{i} \left( \sum_{j=1}^{i} L(v_j) - L(v_0) \right) \leq 1 \)

\( L(v_i) - L(v_{i-1}) \leq 1 \)

\( L(v_i) - L(v_0) \leq i \)
Claim: If $G$ has at least $n$ edges then it has a cycle.

Proof: If $G$ is connected and it has no cycle, it is a tree. But a tree has $n-1$ edges, contradiction.

If not, I can map so that $\# \text{edges} \geq \# \text{vertices}$ (because $\# \text{edges} \geq \# \text{verts}$ in $G$). So by above, it has a cycle.

Claim: If $L(v) = L(u) = i$ and $\{u,v\}$ an edge then there is an odd cycle.

\[ j - i + j - i + 1 = 2(j - i) + 1 \]