CSE 421

Menger's Theorem, Image Segmenation
NP-Completeness

Shayan Oveis Gharan

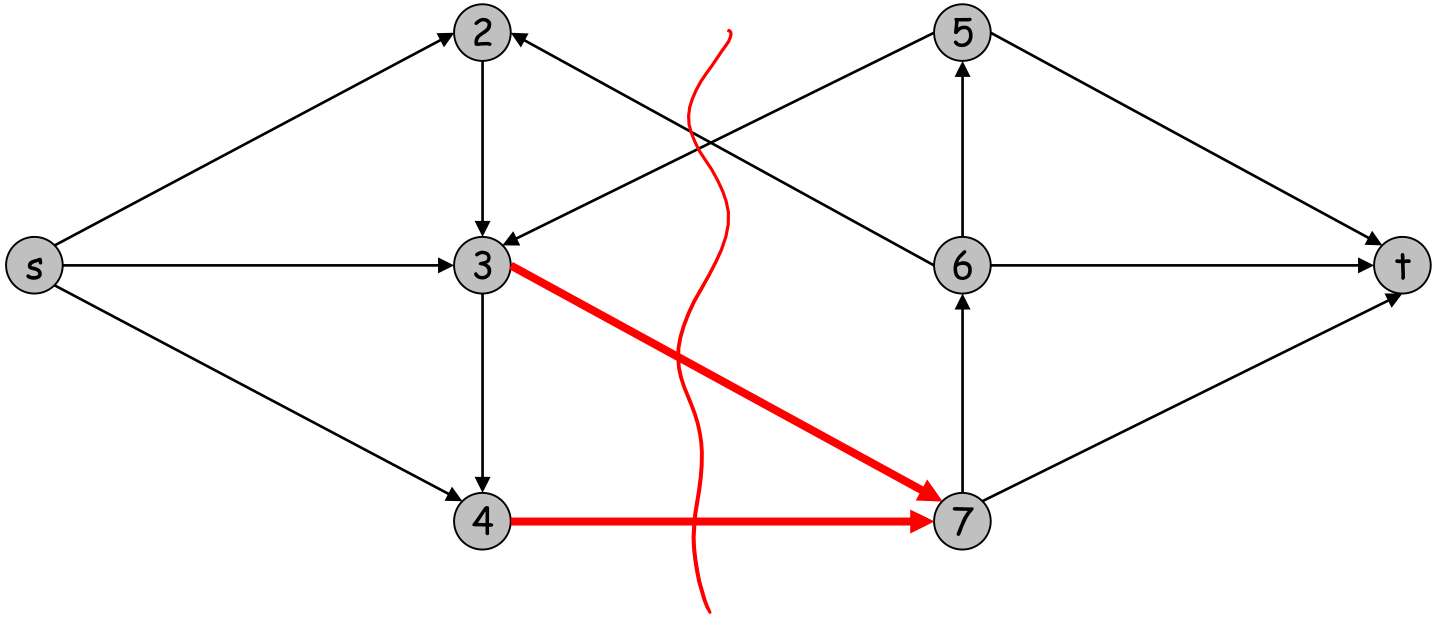
Network Connectivity

Network Connectivity

Given a digraph G = (V, E) and two nodes s and t, find min number of edges whose removal disconnects t from s.

Def. A set of edges $F \subseteq E$ disconnects t from s if all s-t paths uses at least one edge in F.

Ex: In testing network reliability



Network Connectivity using Min Cut

Thm. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf. $\leq$

Suppose the removal of $F \subseteq E$ disconnects t from s, and $|F| = k$.

All s-t paths use at least one edge of F. Hence, the number of edge-disjoint paths is at most k.

Network Connectivity using Min Cut

Thm. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf. ≥

Suppose there are k edge disjoint paths from s to t

So, Max flow is k

So, there is a s-t cut (A,B) s.t.,

cap(A,B)=k

Let F be the edges out of A. So,

|F|=k.

If we remove F we disconnect t from s.

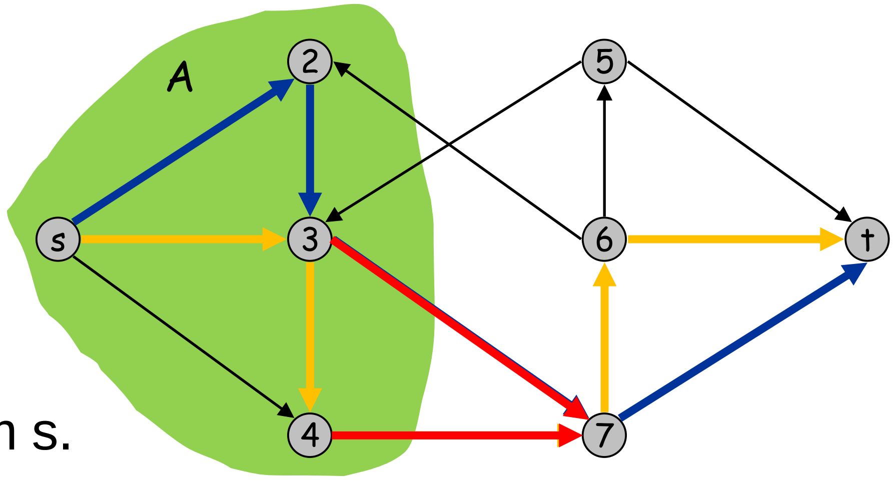


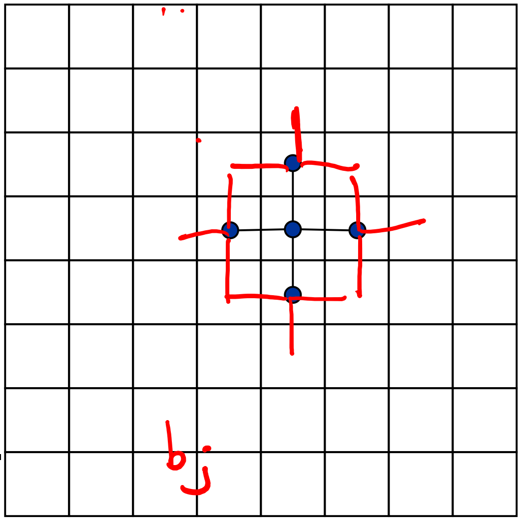
Image Segmentation

Image Segmentation

Given an image we want to separate foreground from background

- Central problem in image processing.
- Divide image into coherent regions.

Foreground / background segmentation



Label each pixel as foreground/background.

- V = set of pixels, E = pairs of neighboring pixels.
- $a_i \geq 0$ is likelihood pixel i in foreground.
- $b_i \geq 0$ is likelihood pixel i in background.
- $p_{i,j} \geq 0$ is separation penalty for labeling one of i and j as foreground, and the other as background.

Goals.

Accuracy: if $a_i > b_i$ in isolation, prefer to label i in foreground.

Smoothness: if many neighbors of i are labeled foreground, we should be inclined to label i as foreground.

Find partition (A, B) that maximizes:

$$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

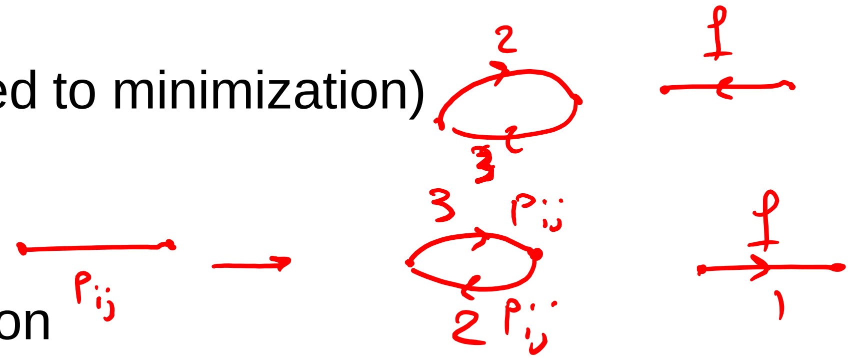
Foreground

Background

Image Seg: Min Cut Formulation

Difficulties:

- Maximization (as opposed to minimization)
- No source or sink
- Undirected graph



Step 1: Turn into Minimization

Maximizing $$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

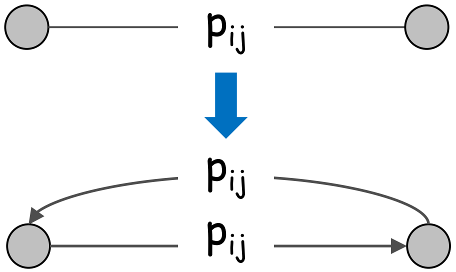
Const

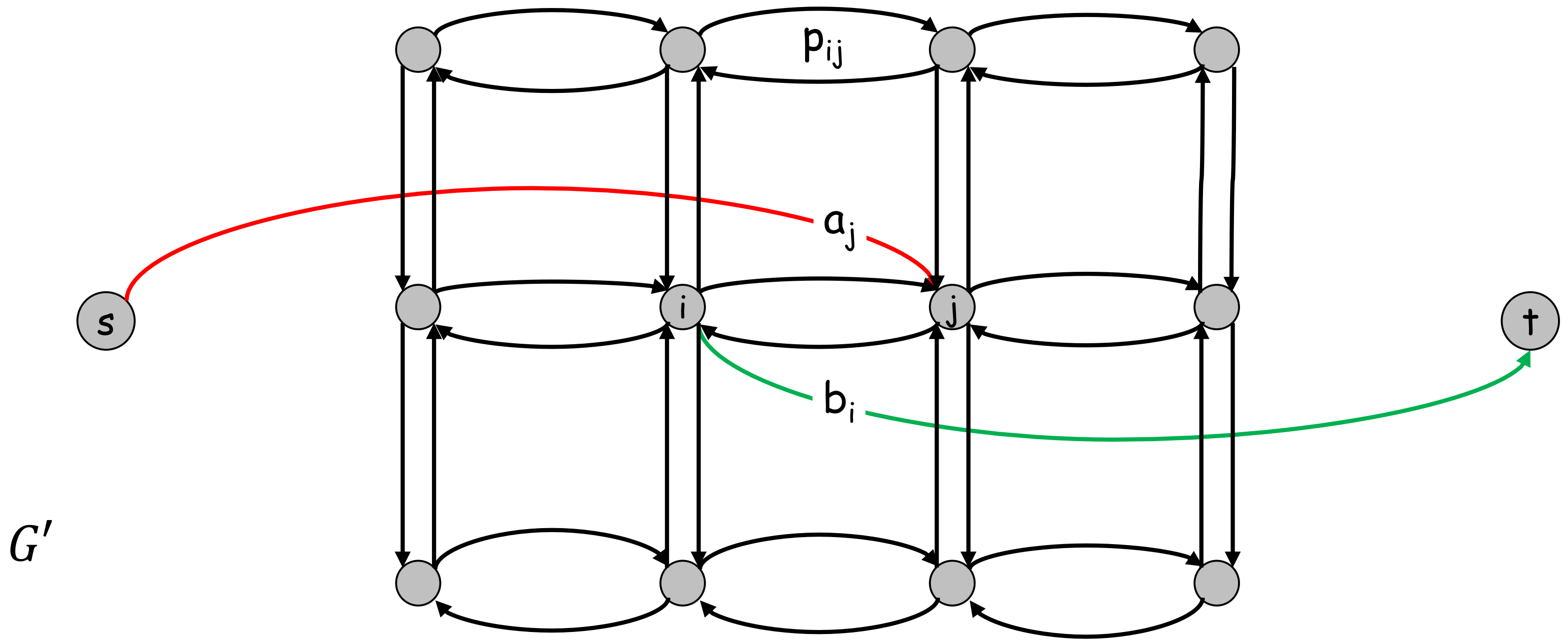
Equivalent to minimizing $$+\sum_{i \in V} a_i + \sum_{j \in V} b_j - \sum_{i \in A} a_i - \sum_{j \in B} b_j + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Equivalent to minimizing $$+\sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Min cut Formulation (cont'd)

G' = (V', E').
Add s to correspond to foreground;
Add t to correspond to background
Use two anti-parallel edges
instead of undirected edge.



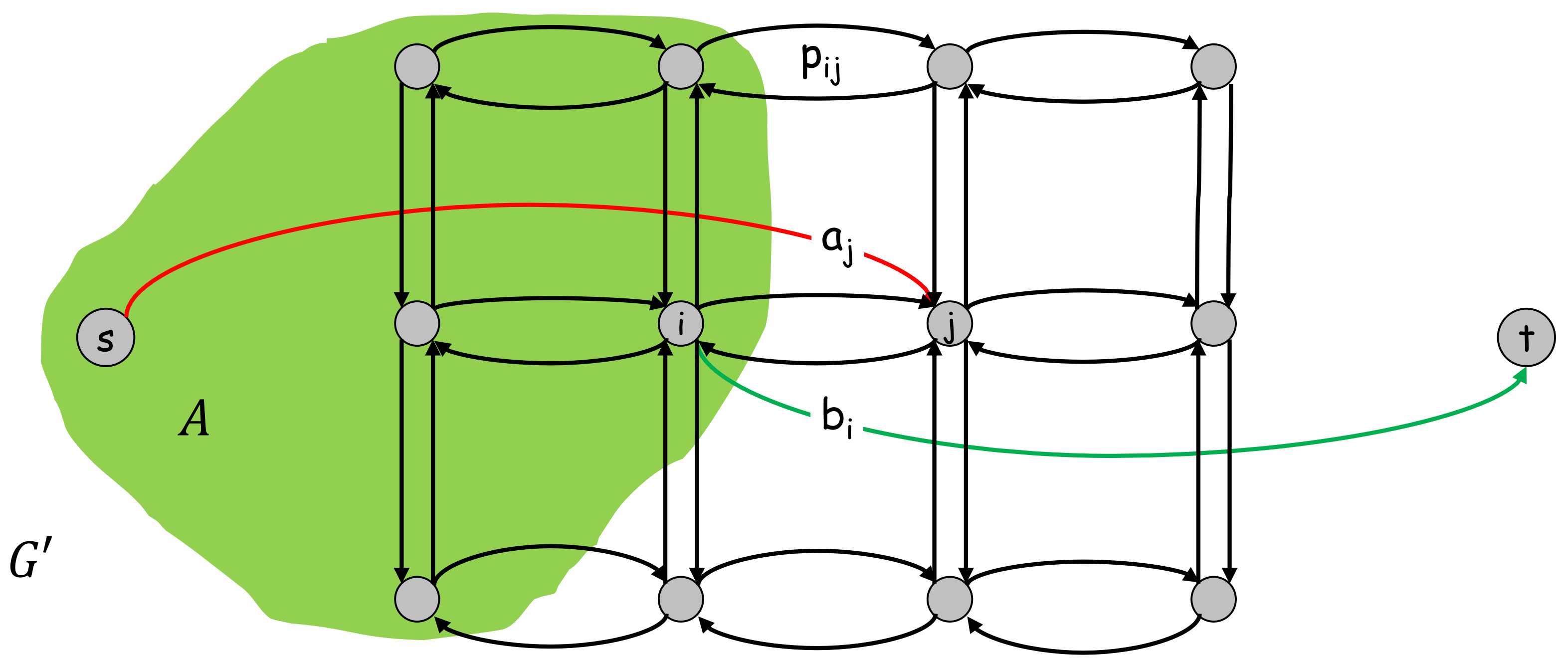


Min cut Formulation (cont'd)

Consider min cut (A, B) in G'. (A = foreground.)

$$cap(A,B) = \sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Precisely the quantity we want to minimize.



Reductions & NP-Completeness

Computational Complexity

Goal: Classify problems according to the amount of computational resources used by the best algorithms that solve them

Here we focus on time complexity

Recall: worst-case running time of an algorithm

- **max** # steps algorithm takes on any input of size **n**

Computational Complexity and Reduction

In most cases, we cannot characterize the true hardness of a computational problem

So?

We only reduce the number of problems

Want to be able to make statements of the for

- "If we could solve problem B in polynomial time then we can solve problem A in polynomial time"
- "Problem B is at least as hard as problem A"

Polynomial Time Reduction

Def A $\leq_P$ B: if there is an algorithm for problem A using a 'black box' (subroutine) that solve problem B s.t.,

- Algorithm uses only a polynomial number of steps
- Makes only a polynomial number of calls to a subroutine for **B**

So,

B is Polynomial time solvable ⇒ A is Polynomial time solvable

Conversely,

No efficient Algorithm for A ⇒ No efficient Algorithm for B

In words, B is as hard as A (it can be even harder)

$\leq_p^1$ Reductions

Here, We will always use a restricted form of polynomial-time reduction often called Karp or many-to-one reduction

$A \leq_p^1 B$: if and only if there is an algorithm for A given a black box solving B that on input x

- Runs for polynomial time computing an input f(x) of B
- Makes one call to the black box for B for input f(x)
- Returns the answer that the black box gave

We say that the function f(.) is the reduction

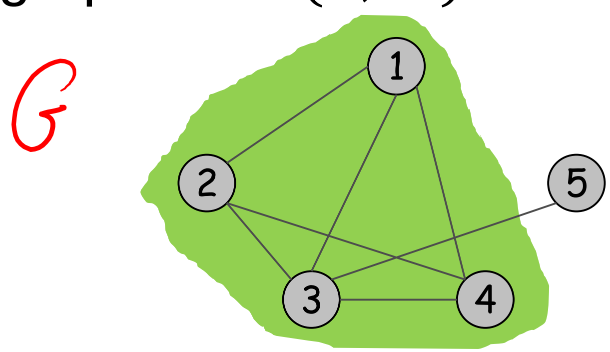
Example 1: Indep Set $\le_p$ Clique

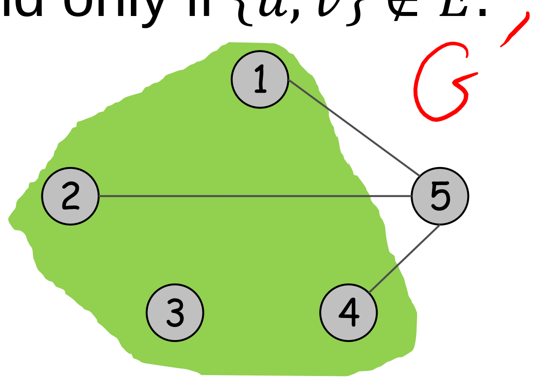
Indep Set: Given G=(V,E) and an integer k, is there $S \subseteq V$ s.t. $|S| \geq k$ an no two vertices in S are joined by an edge?

Clique: Given a graph G=(V,E) and an integer k, is there $S \subseteq V$, |U| $\geq$ k s.t., every pair of vertices in S is joined by an edge?

Claim: Indep Set $\le_p$ Clique

Pf: Given $G = (V, E)$ and instance of indep Set. Construct a new graph $G' = (V, E')$ where $\{u, v\} \in E'$ if and only if $\{u, v\} \notin E$.





S is an independ set in G

S is an Clique in G'

Example 2: Vertex Cover $\leq_p$ Indep Set

Vertex Cover: Given a graph G=(V,E) and an integer k, is there a vertex cover of size at most k?

Claim: For any graph $G = (V, E)$, S is an independent set iff $V - S$ is a vertex cover

Pf:

=> Let S be a independent set of G

Then, S has at most one endpoint of every edge of G

So, $V - S$ has at least one endpoint of every edge of G

So, $V - S$ is a vertex cover.

<= Suppose $V - S$ is a vertex cover

Then, there is no edge between vertices of S (otherwise, $V - S$ is not a vertex cover)

So, S is an independent set.

Example 3: Vertex Cover $\leq_p$ Set Cover

Set Cover: Given a set U, collection of subsets $S_1, \ldots, S_m$ of U and an integer k, is there a collection of k sets that contain all elements of U?

Claim: Vertex Cover $\leq_p$ Set Cover

Pf:

Given $(G = (V, E), k)$ of vertex cover we construct a set cover input $f(G, k)$

- $U = E$
- For each $v \in V$ we create a set S_v of all edges connected to v

This clearly is a polynomial-time reduction

So, we need to prove it gives the right answer

Example 3: Vertex Cover $\leq_p$ Set Cover

Claim: Vertex Cover $\leq_p$ Set Cover

Pf: Given $(G = (V, E), k)$ of vertex cover we construct a set cover input $f(G, k)$

- $U = E$
- For each $v \in V$ we create a set S_v of all edges connected to v

Vertex-Cover (G,k) is yes => Set-Cover f(G,k) is yes

If a set $W \subseteq V$ covers all edges,, just choose S_v for all $v \in W$, it covers all U.

Set-Cover f(G,k) is yes => Vertex-Cover (G,k) is yes

If $(S_{v_1}, \ldots, S_{v_k})$ covers all U, the set $\{v_1, \ldots, v_k\}$ covers all edges of G.

Decision Problems

A decision problem is a computational problem where the answer is just yes/no

Here, we study computational complexity of decision Problems.

Why?

- much simpler to deal with
- Decision version is not harder than Search version, so it is easier to lower bound Decision version
- Less important, usually, you can use decider multiple times to find an answer .