Residual Graph

Original edge: \( e = (u, v) \in E \).
- Flow \( f(e) \), capacity \( c(e) \).

Residual edge.
- "Undo" flow sent.
- \( e = (u, v) \) and \( e^R = (v, u) \).
- Residual capacity:
  \[
  c_f(e) = \begin{cases} 
  c(e) - f(e) & \text{if } e \in E \\
  f(e) & \text{if } e^R \in E 
  \end{cases}
  \]

Residual graph: \( G_f = (V, E_f) \).
- Residual edges with positive residual capacity.
- \( E_f = \{ e : f(e) < c(e) \} \cup \{ e : f(e^R) > 0 \} \).
Augmenting Path Algorithm

Augment(f, c, P) {
    b <- bottleneck(P)
    foreach e £ P {
        if (e £ E) f(e) <- f(e) + b
        else f(e^R) <- f(e) - b
    }
    return f
}

Ford-Fulkerson(G, s, t, c) {
    foreach e £ E  f(e) <- 0
    G_f <- residual graph

    while (there exists augmenting path P) {
        f <- Augment(f, c, P)
        update G_f
    }
    return f
}
Max Flow Min Cut Theorem

**Augmenting path theorem.** Flow \( f \) is a max flow iff there are no augmenting paths.

**Max-flow min-cut theorem.** [Ford-Fulkerson 1956] The value of the max s-t flow is equal to the value of the min s-t cut.

**Proof strategy.** We prove both simultaneously by showing the TFAE:

1. There exists a cut \((A, B)\) such that \( v(f) = \text{cap}(A, B) \).
2. Flow \( f \) is a max flow.
3. There is no augmenting path relative to \( f \).

(i) \( \Rightarrow \) (ii) This was the corollary to weak duality lemma.

(ii) \( \Rightarrow \) (iii) We show contrapositive.

Let \( f \) be a flow. If there exists an augmenting path, then we can improve \( f \) by sending flow along that path.
Pf of Max Flow Min Cut Theorem

(iii) => (i)
No augmenting path for f => there is a cut (A,B): v(f)=cap(A,B)

• Let f be a flow with no augmenting paths.
• Let A be set of vertices reachable from s in residual graph.
• By definition of A, s ∈ A.
• By definition of f, t ∉ A.

\[ v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \]

\[ = \sum_{e \text{ out of } A} c(e) \]

\[ = \text{cap}(A, B) \]
Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value $f(e)$ and every residual capacities $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \leq nC$ iterations, if $f^*$ is optimal flow.

Pf. Each augmentation increase value by at least 1.

Corollary. If C = 1, Ford-Fulkerson runs in O(mn) time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer.

Pf. Since algorithm terminates, theorem follows from invariant.
Applications of Max Flow:
Bipartite Matching
Maximum Matching Problem

Given an undirected graph $G = (V, E)$. A set $M \subseteq E$ is a matching if each node appears in at most one edge in $M$.

**Goal**: find a matching with largest cardinality.
Bipartite Matching Problem

Given an undirected bipartite graph \( G = (X \cup Y, E) \)
A set \( M \subseteq E \) is a matching if each node appears in at most one edge in \( M \).
**Goal**: find a matching with largest cardinality.
Bipartite Matching using Max Flow

Create digraph $H$ as follows:
- Orient all edges from $X$ to $Y$, and assign infinite (or unit) capacity.
- Add source $s$, and unit capacity edges from $s$ to each node in $L$.
- Add sink $t$, and unit capacity edges from each node in $R$ to $t$. 
Bipartite Matching: Proof of Correctness

**Thm.** Max cardinality matching in $G =$ value of max flow in $H$.

**Pf.** $\leq$

Given max matching $M$ of cardinality $k$.
Consider flow $f$ that sends 1 unit along each of $k$ edges of $M$.
f is a flow, and has cardinality $k$. ▪
**Bipartite Matching: Proof of Correctness**

**Thm.** Max cardinality matching in $G = \text{value of max flow in } H$.

**Pf. (of $\geq$)** Let $f$ be a max flow in $H$ of value $k$.

Integrality theorem $\Rightarrow$ $k$ is integral and we can assume $f$ is 0-1.

Consider $M = \text{set of edges from } X \text{ to } Y \text{ with } f(e) = 1$.

- each node in $X$ and $Y$ participates in at most one edge in $M$
- $|M| = k$: consider $s$-$t$ cut $(s \cup X, t \cup Y)$
Perfect Bipartite Matching
Perfect Bipartite Matching

Def. A matching $M \subseteq E$ is perfect if each node appears in exactly one edge in $M$.

Q. When does a bipartite graph have a perfect matching?

Structure of bipartite graphs with perfect matchings:
- Clearly we must have $|X| = |Y|$.
- What other conditions are necessary?
- What conditions are sufficient?
Perfect Bipartite Matching: \( N(S) \)

**Def.** Let \( S \) be a subset of nodes, and let \( N(S) \) be the set of nodes adjacent to nodes in \( S \).

**Observation.** If a bipartite graph \( G \) has a perfect matching, then \( |N(S)| \geq |S| \) for all subsets \( S \subseteq X \).

**Pf.** Each \( v \in S \) has to be matched to a unique node in \( N(S) \).
Marriage Theorem

Thm: [Frobenius 1917, Hall 1935] Let $G = (X \cup Y, E)$ be a bipartite graph with $|X| = |Y|$. Then, $G$ has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq X$.

Pf. $\Rightarrow$
This was the previous observation. If $|N(S)| < |S|$ for some $S$, then there is no perfect matching.
Marriage Theorem

Pf. \( \exists S \subseteq X \) s.t., \( |N(S)| < |S| \) \( \iff \) G does not a perfect matching
Formulate as a max-flow and let \((A, B)\) be the min s-t cut
G has no perfect matching \( \Rightarrow \) \( v(f^*) < |X| \). So, \( cap(A, B) < |X| \)
Define \( X_A = X \cap A, X_B = X \cap B, Y_A = Y \cap A \)
Then, \( cap(A, B) \geq |X_B| + |Y_A| \)
Since min-cut does not use \( \infty \) edges, \( N(X_A) \subseteq Y_A \)
\( |N(X_A)| \leq |Y_A| \leq cap(A, B) - |X_B| = cap(A, B) - |X| + |X_A| < |X_A| \)
Bipartite Matching Running Time

Which max flow algorithm to use for bipartite matching?

Generic augmenting path: \( O(m \text{ val}(f^*)) = O(mn) \).
Capacity scaling: \( O(m^2 \log C) = O(m^2) \).
Shortest augmenting path: \( O(m n^{1/2}) \).

Non-bipartite matching.

Structure of non-bipartite graphs is more complicated, but well-understood. \([\text{Tutte-Berge, Edmonds-Galai}]\)
Blossom algorithm: \( O(n^4) \). \([\text{Edmonds 1965}]\)
Best known: \( O(m n^{1/2}) \). \([\text{Micali-Vazirani 1980}]\)