CSE 421

Set Cover, Dynamic Programming

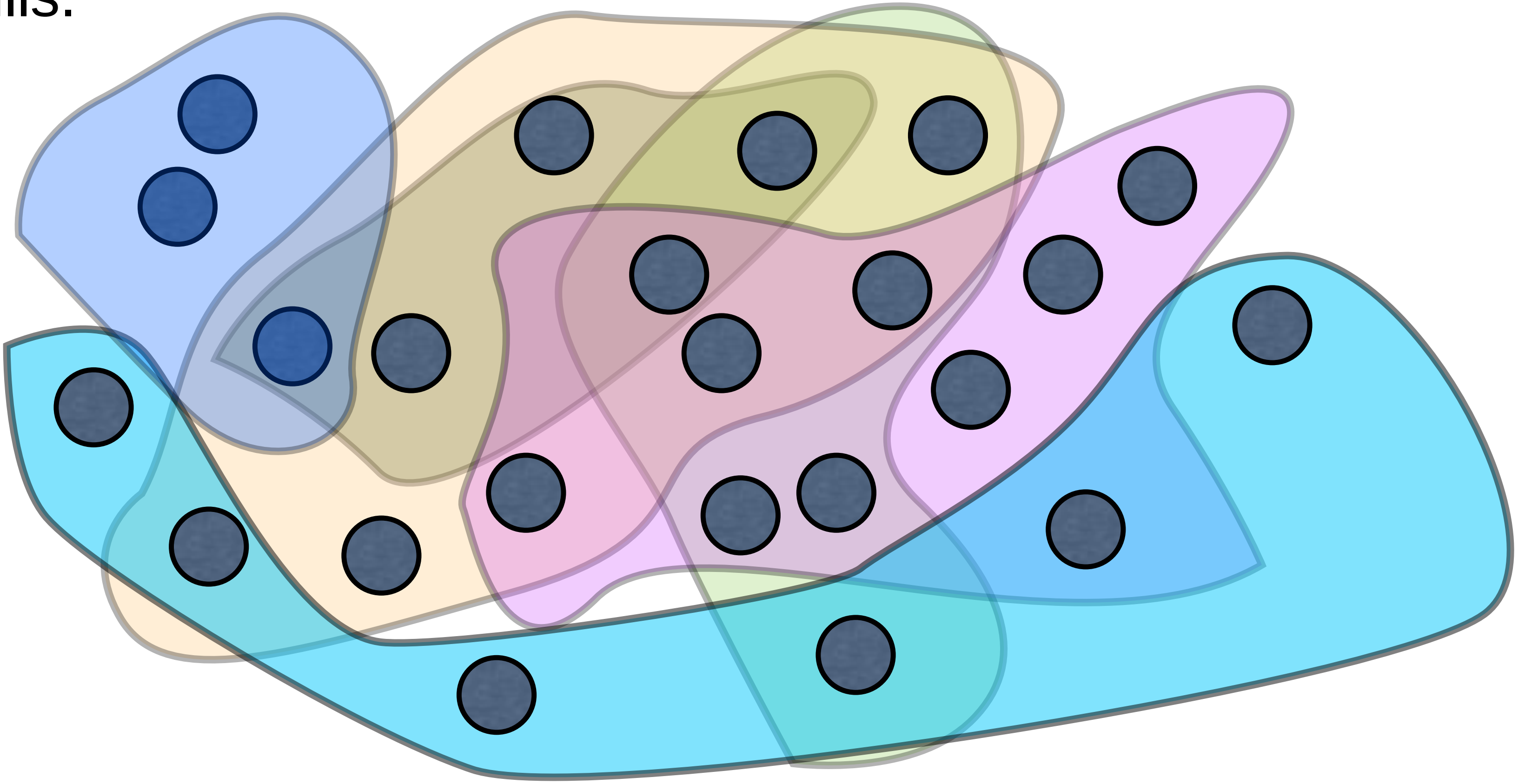
Shayan Oveis Gharan

Set Cover

Given a number of sets on a ground set of elements,

Goal: choose minimum number of sets that cover all.

e.g., a company wants to hire employees with certain skills.

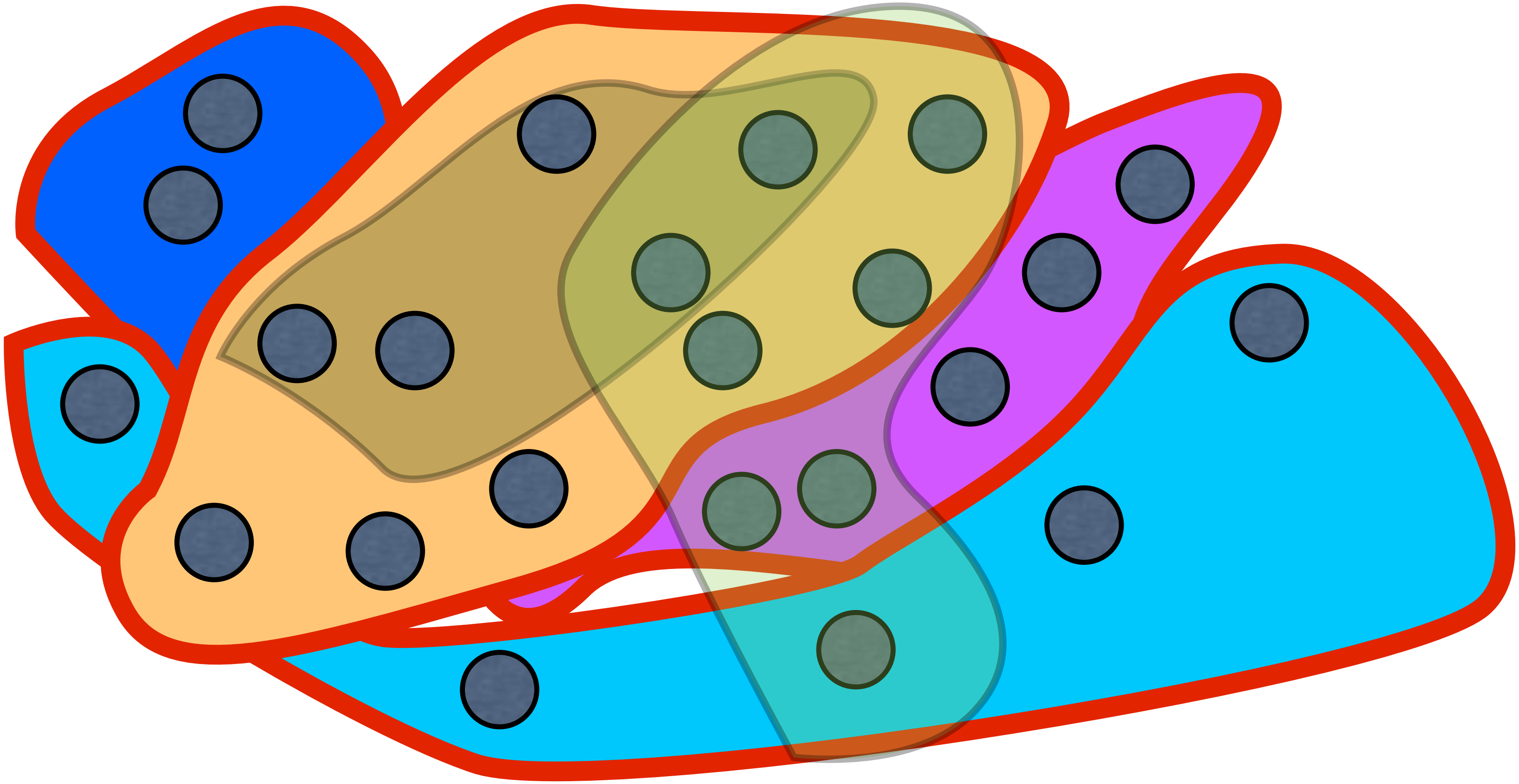


Set Cover

Given a number of sets on a ground set of elements,

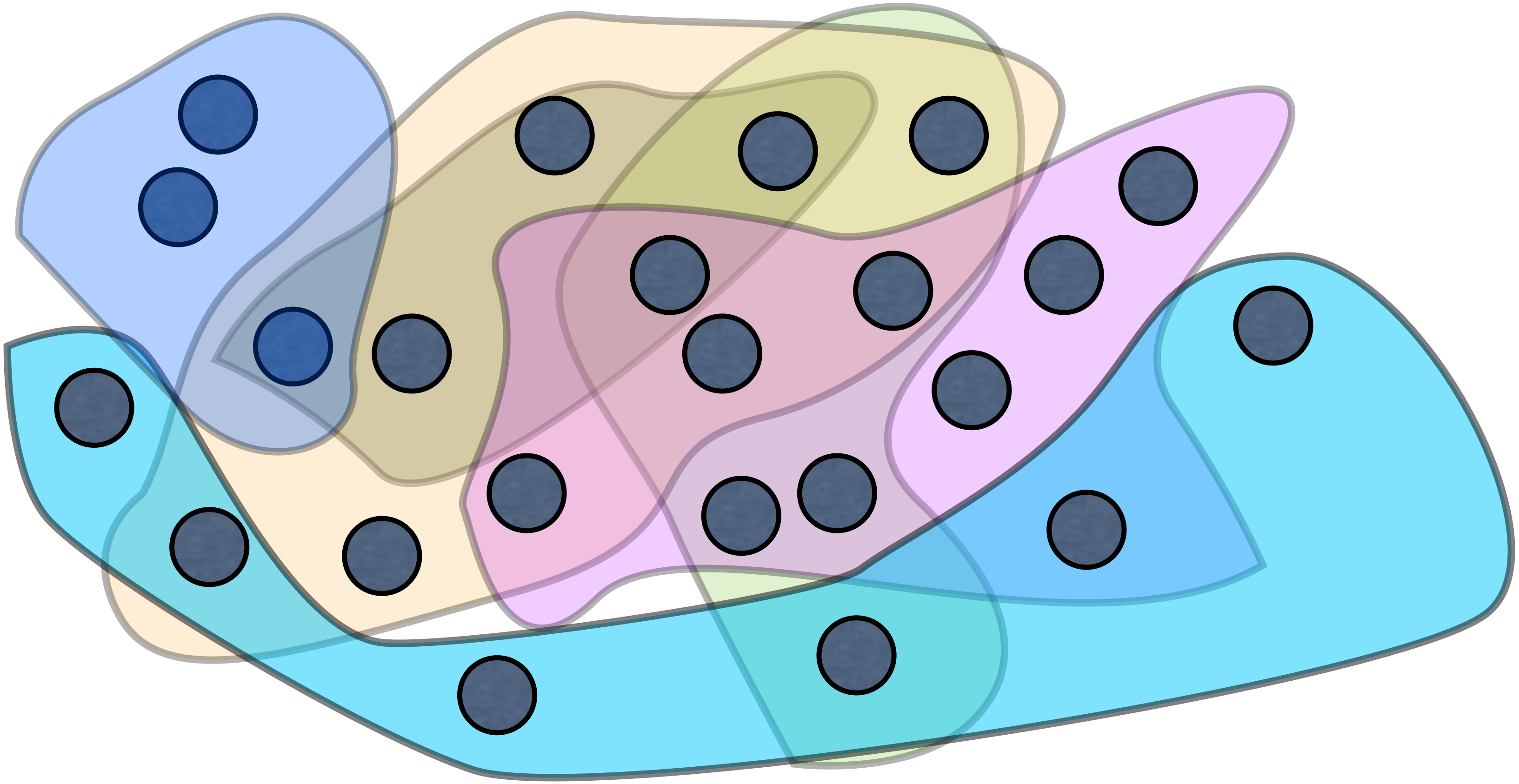
Goal: choose minimum number of sets that cover all.

Set cover = 4



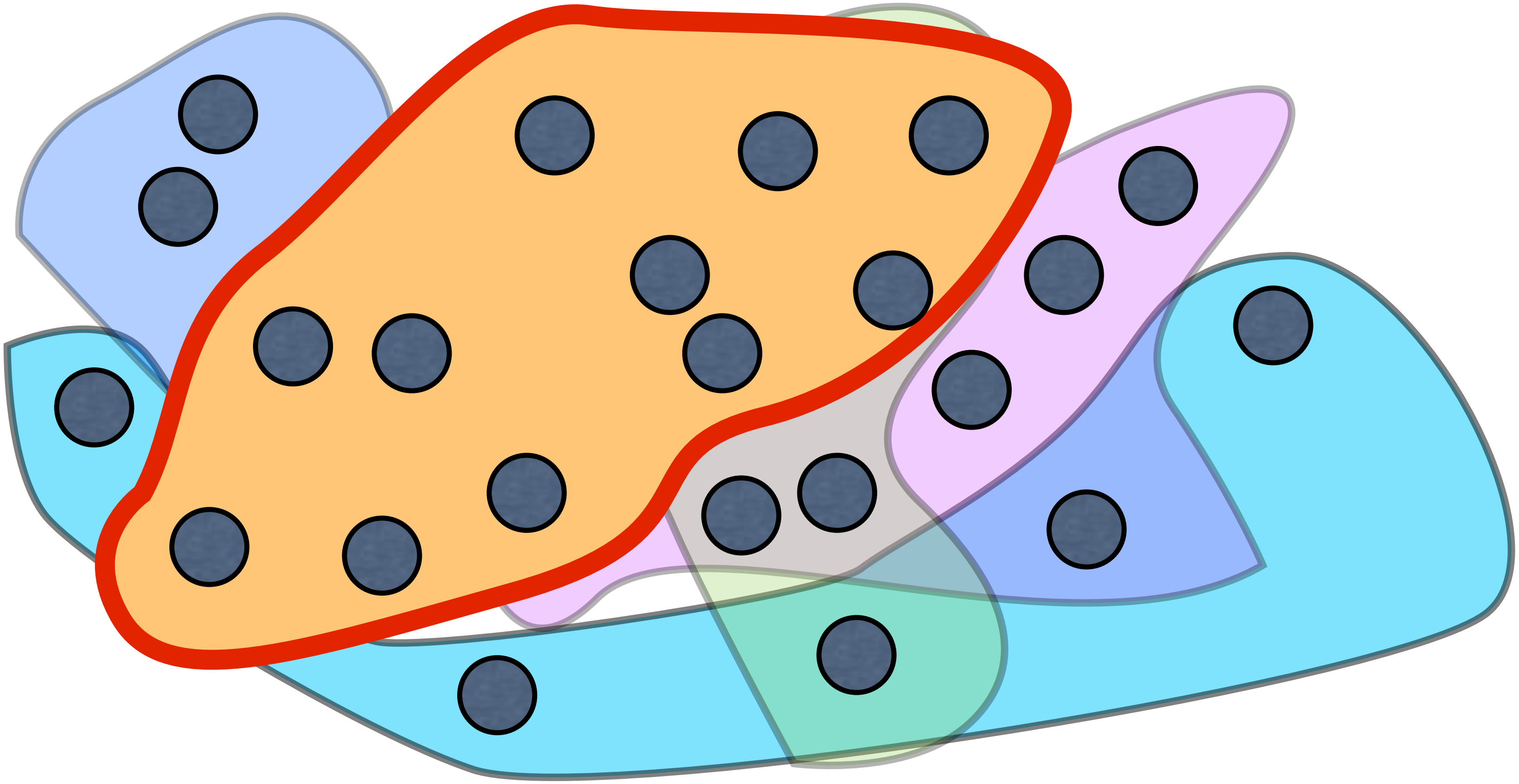
A Greedy Algorithm

Strategy: Pick the set that maximizes # new elements covered



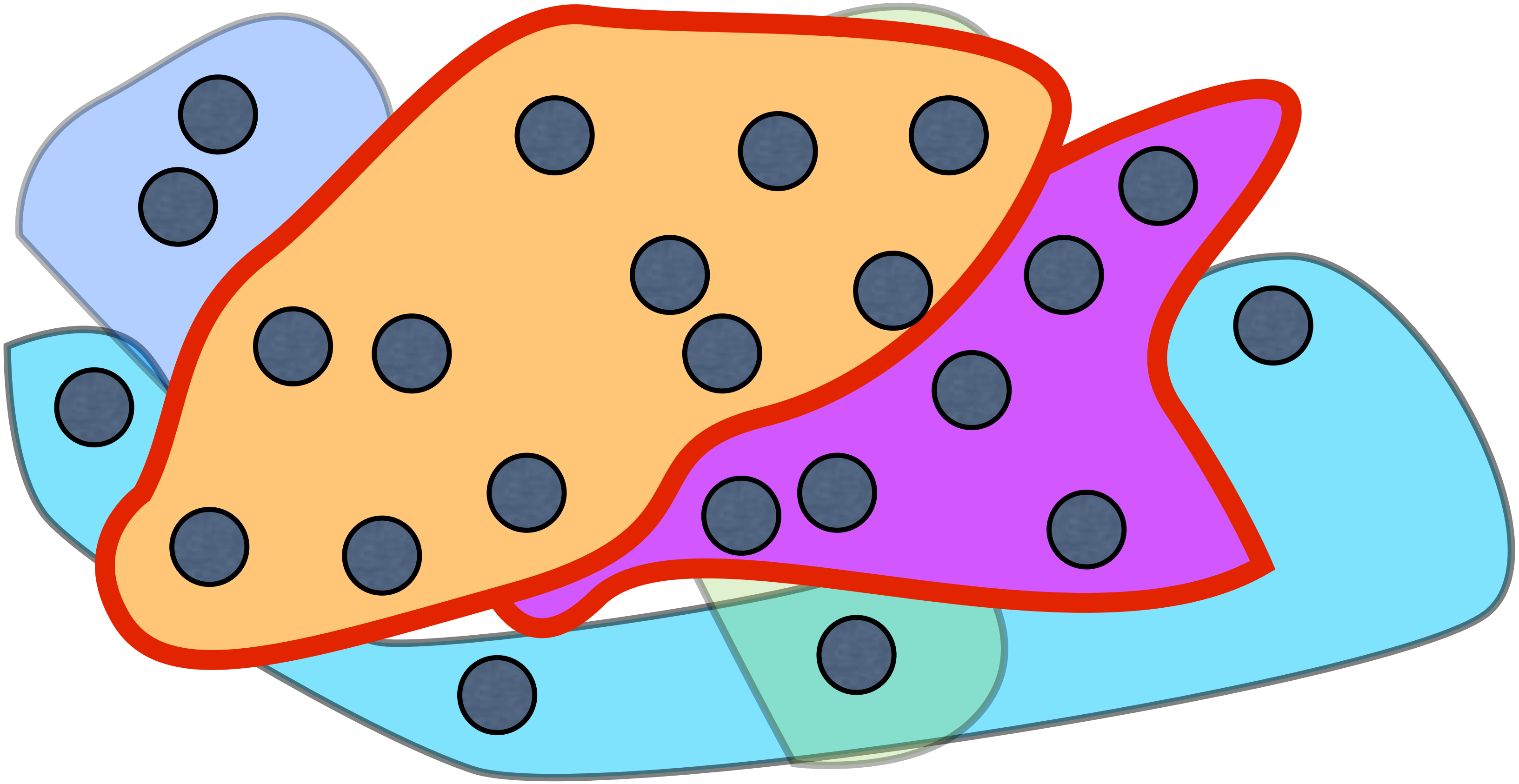
A Greedy Algorithm

Strategy: Pick the set that maximizes # new elements covered



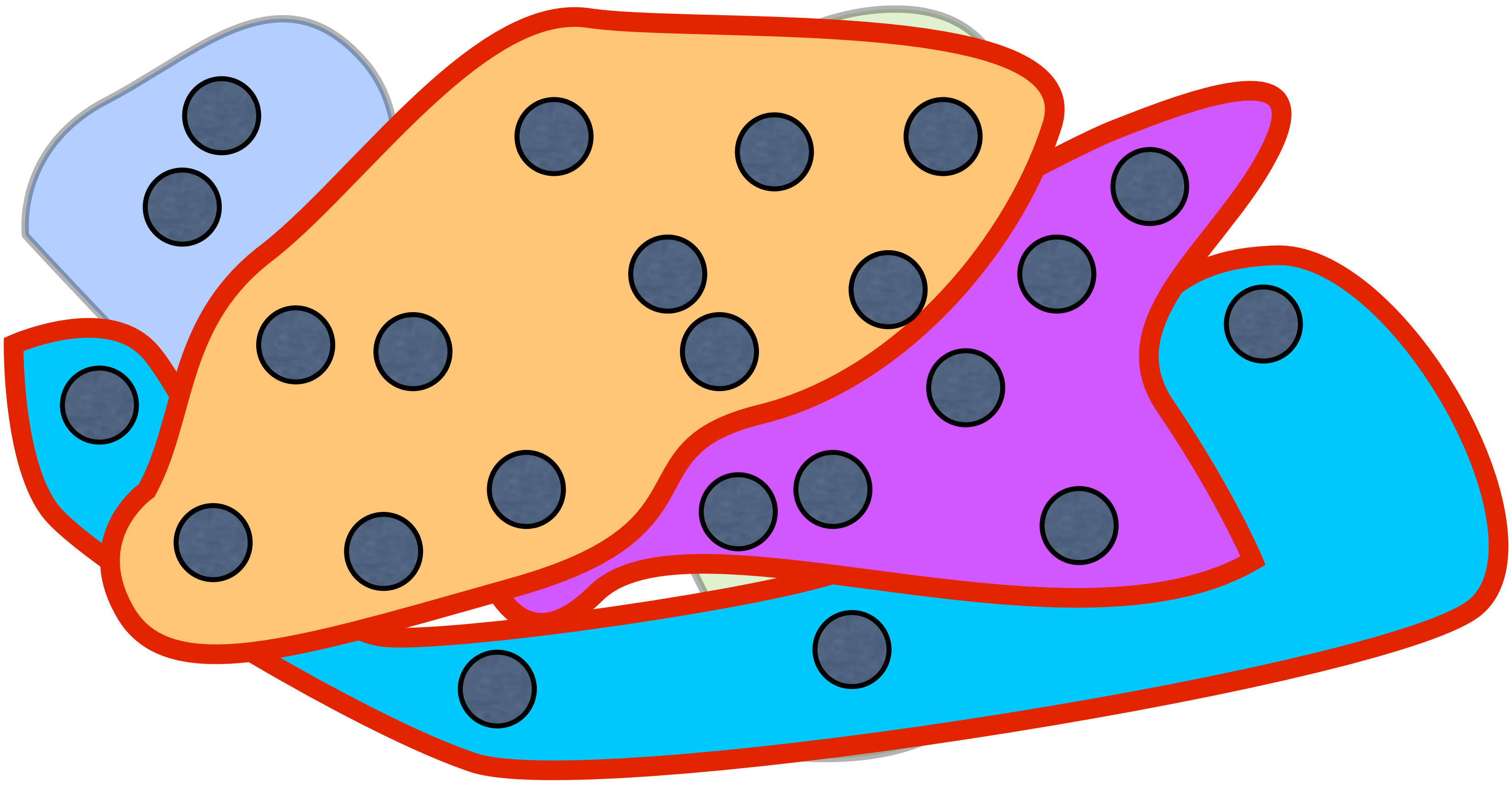
A Greedy Algorithm

Strategy: Pick the set that maximizes # new elements covered



A Greedy Algorithm

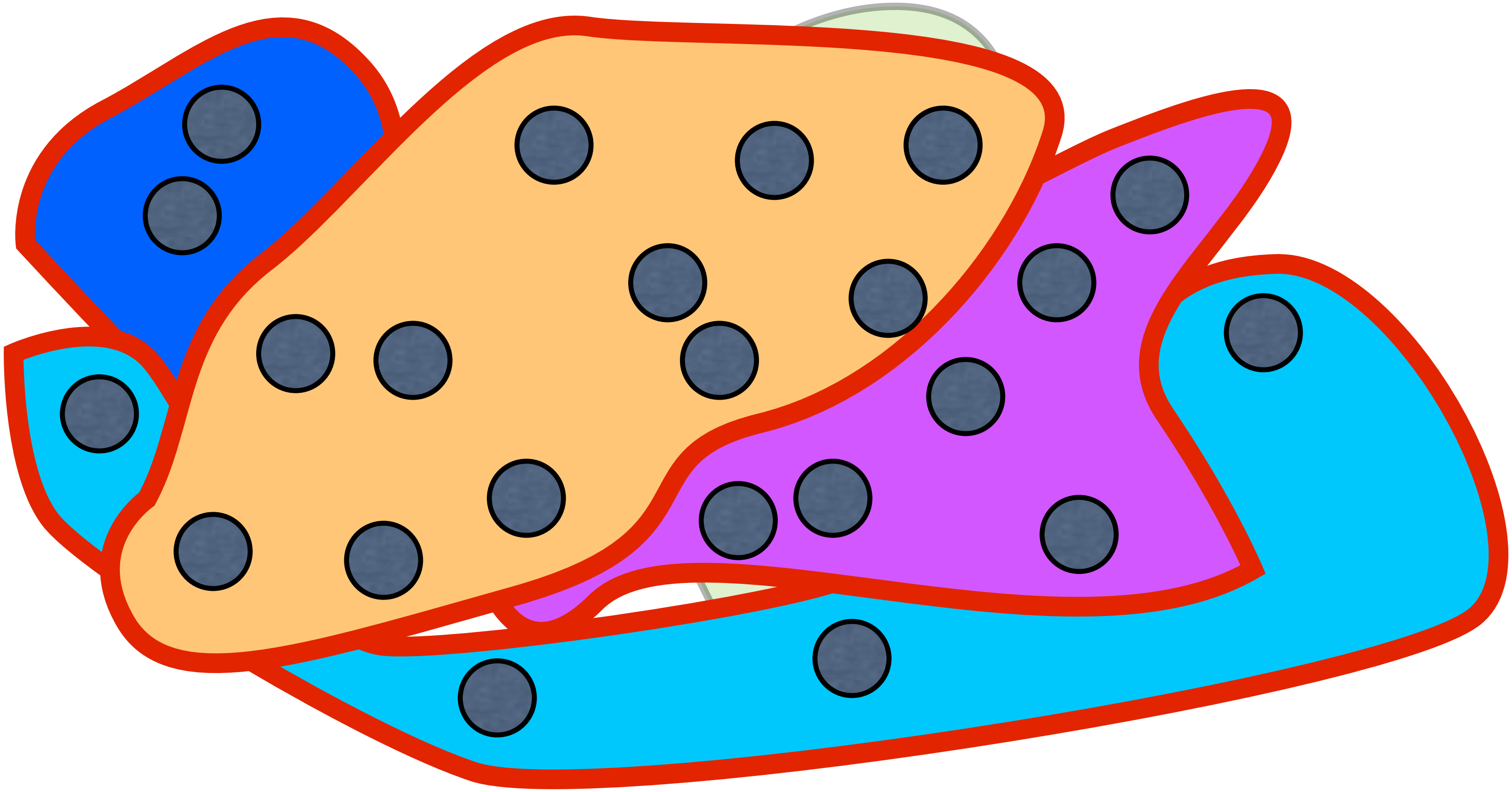
Strategy: Pick the set that maximizes # new elements covered



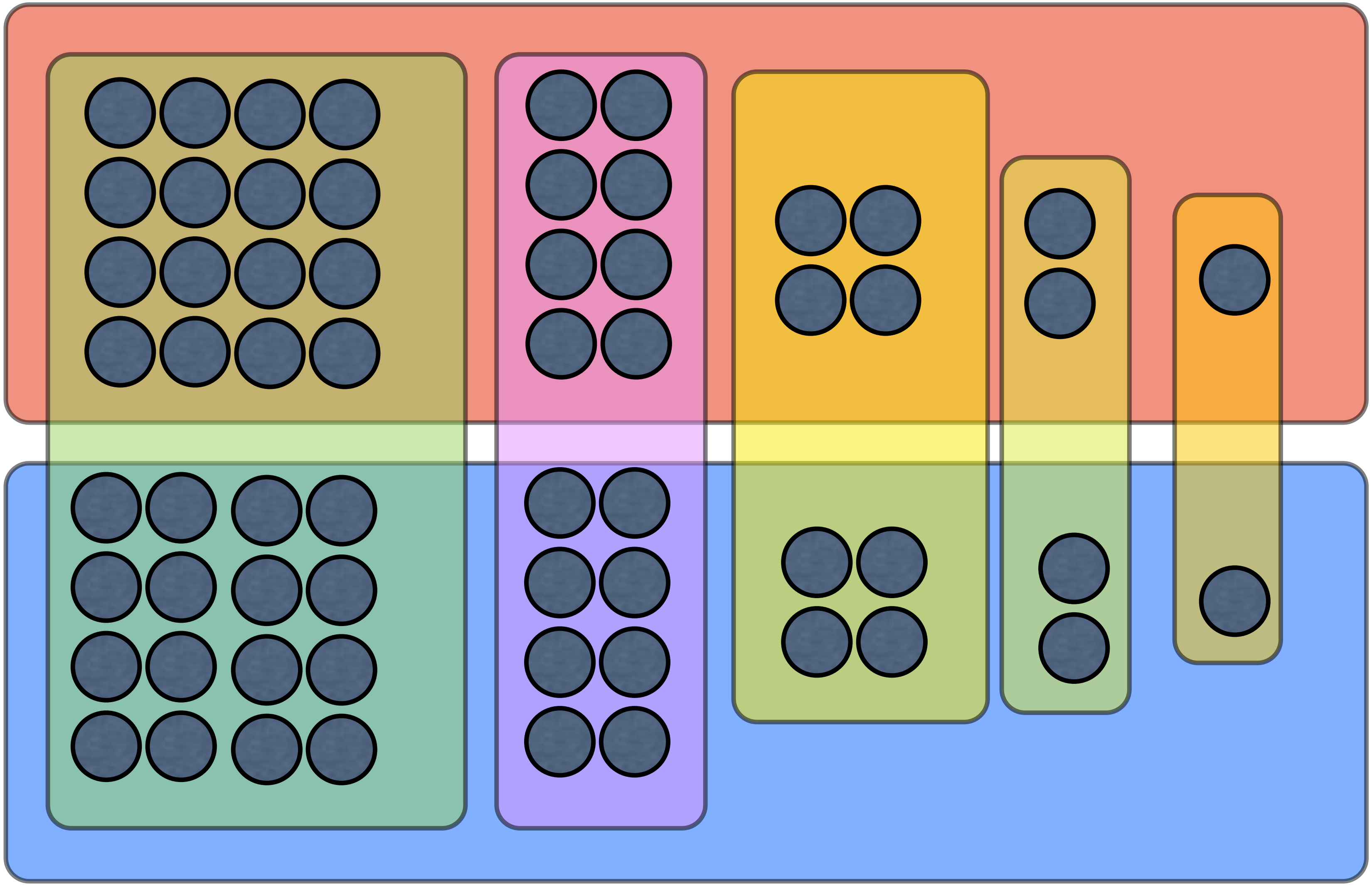
A Greedy Algorithm

Strategy: Pick the set that maximizes # new elements covered

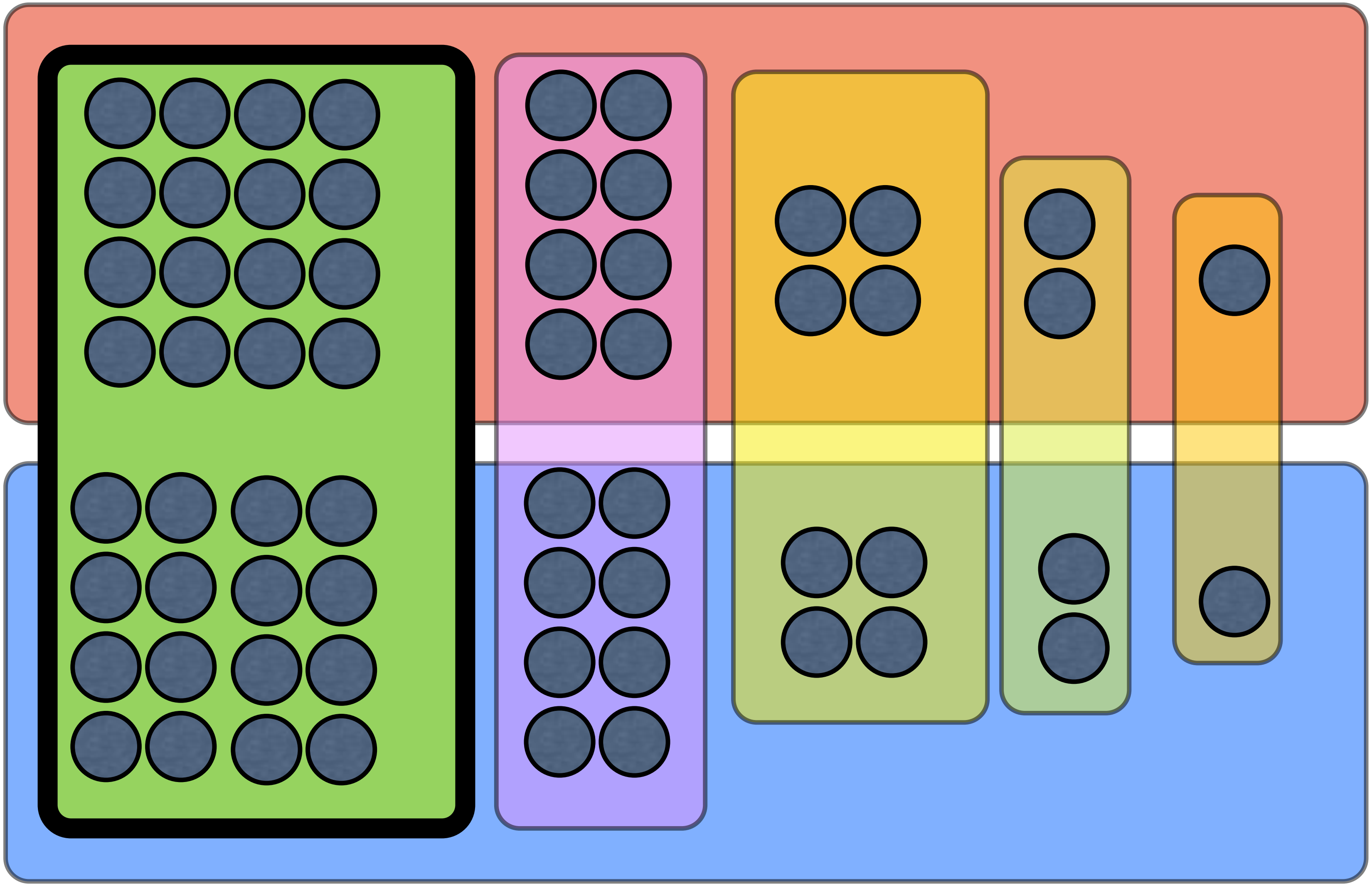
Thm: Greedy has ln n approximation ratio



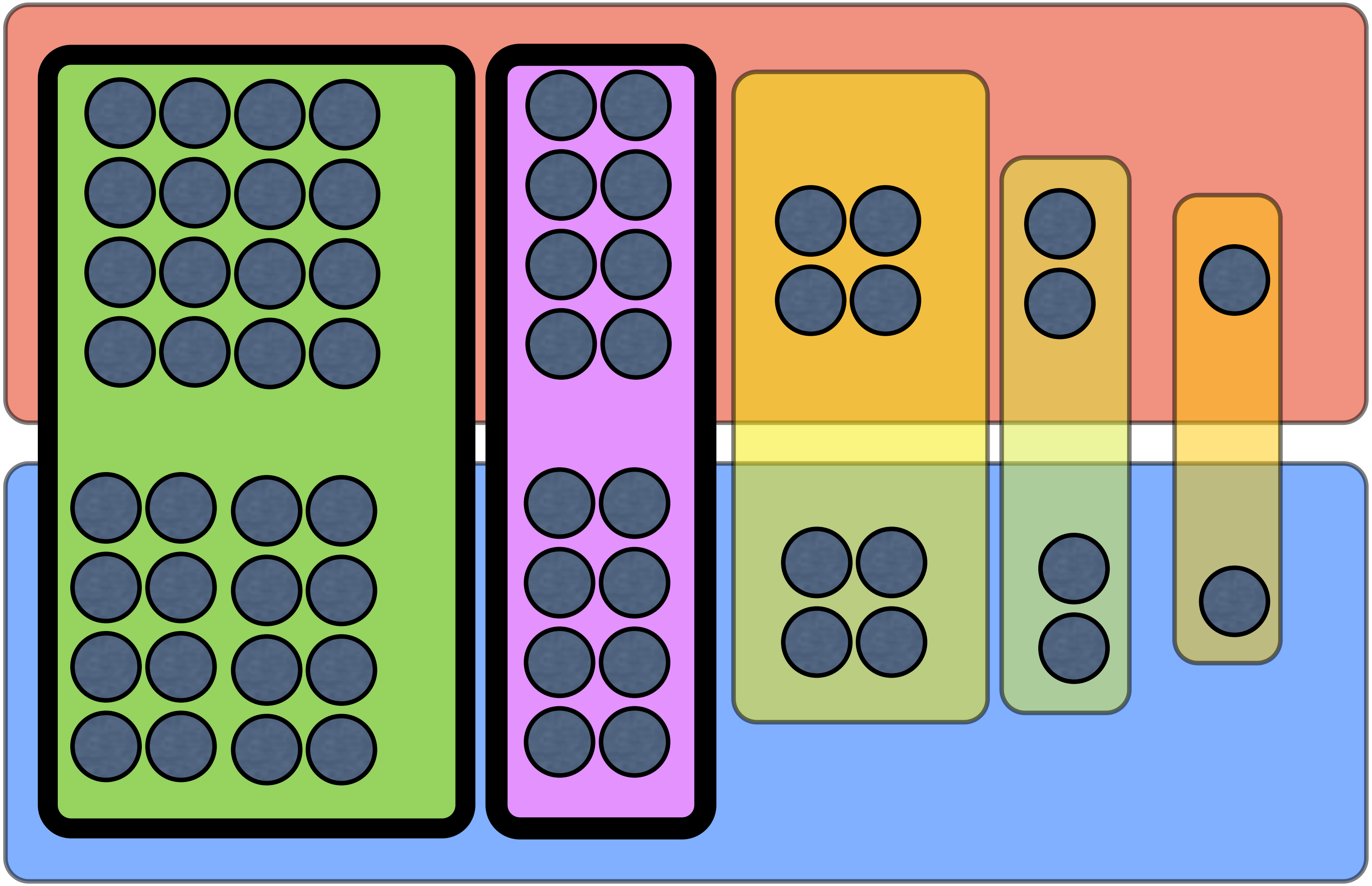
A Tight Example for Greedy



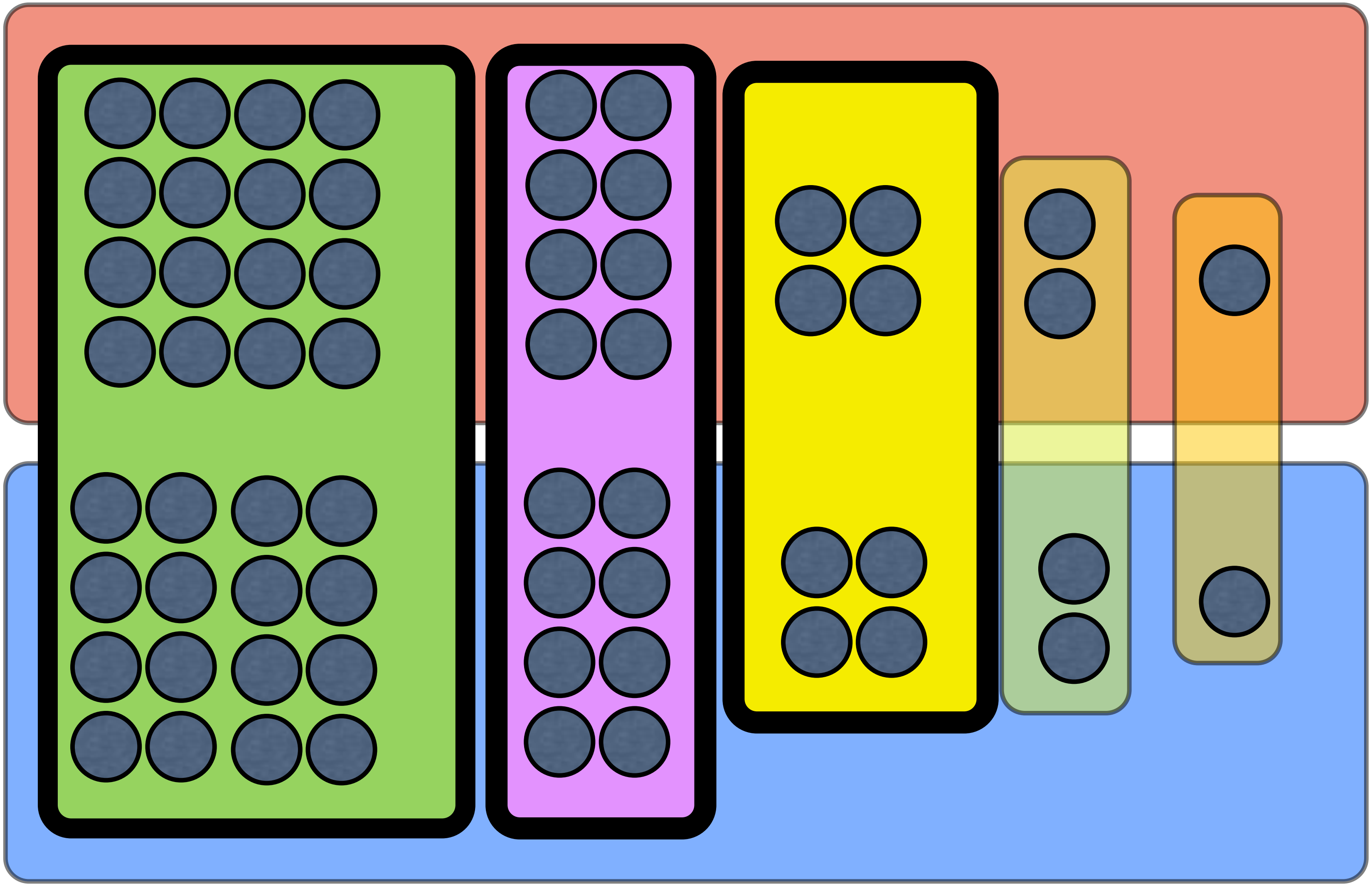
A Tight Example for Greedy



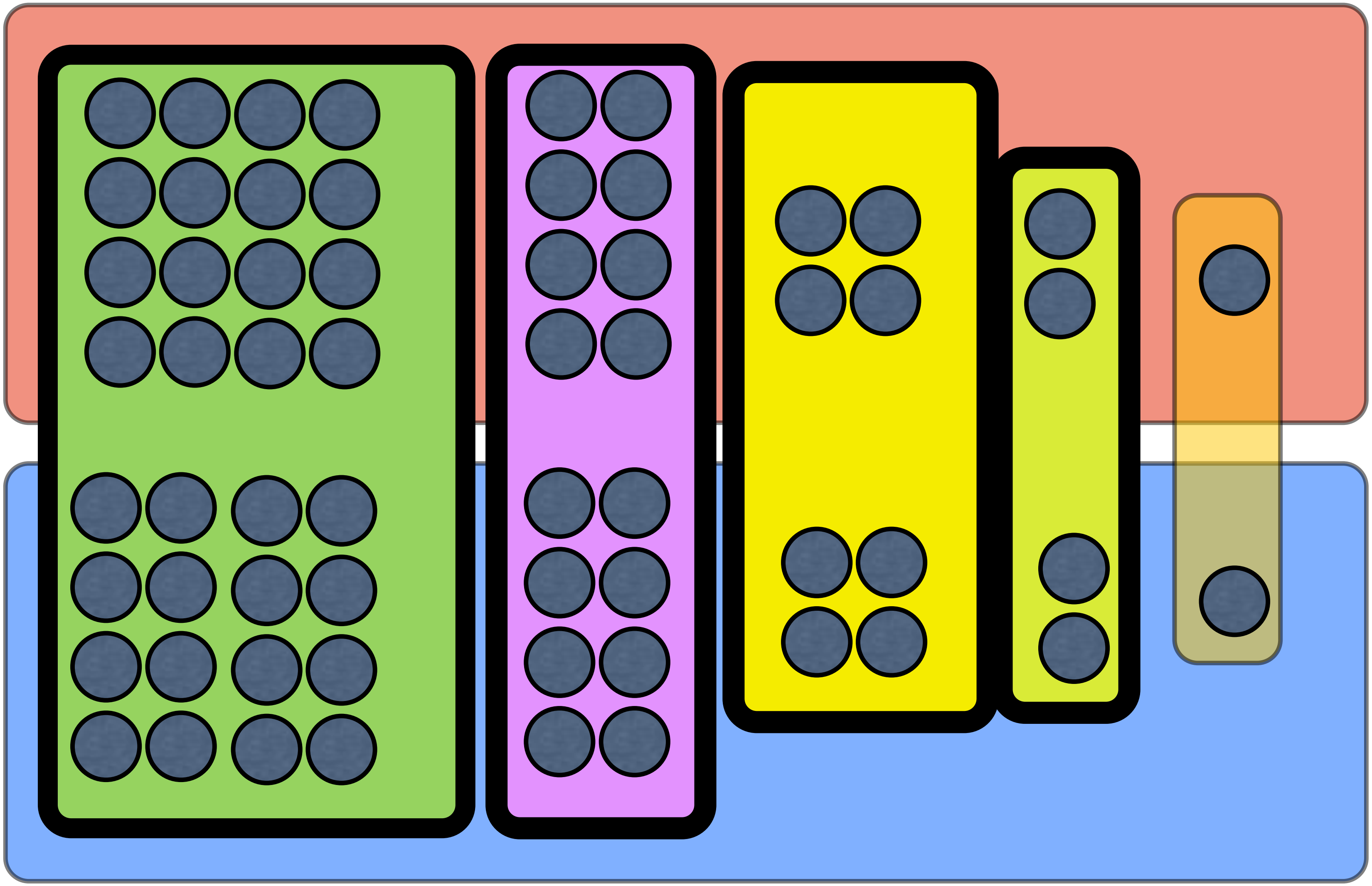
A Tight Example for Greedy



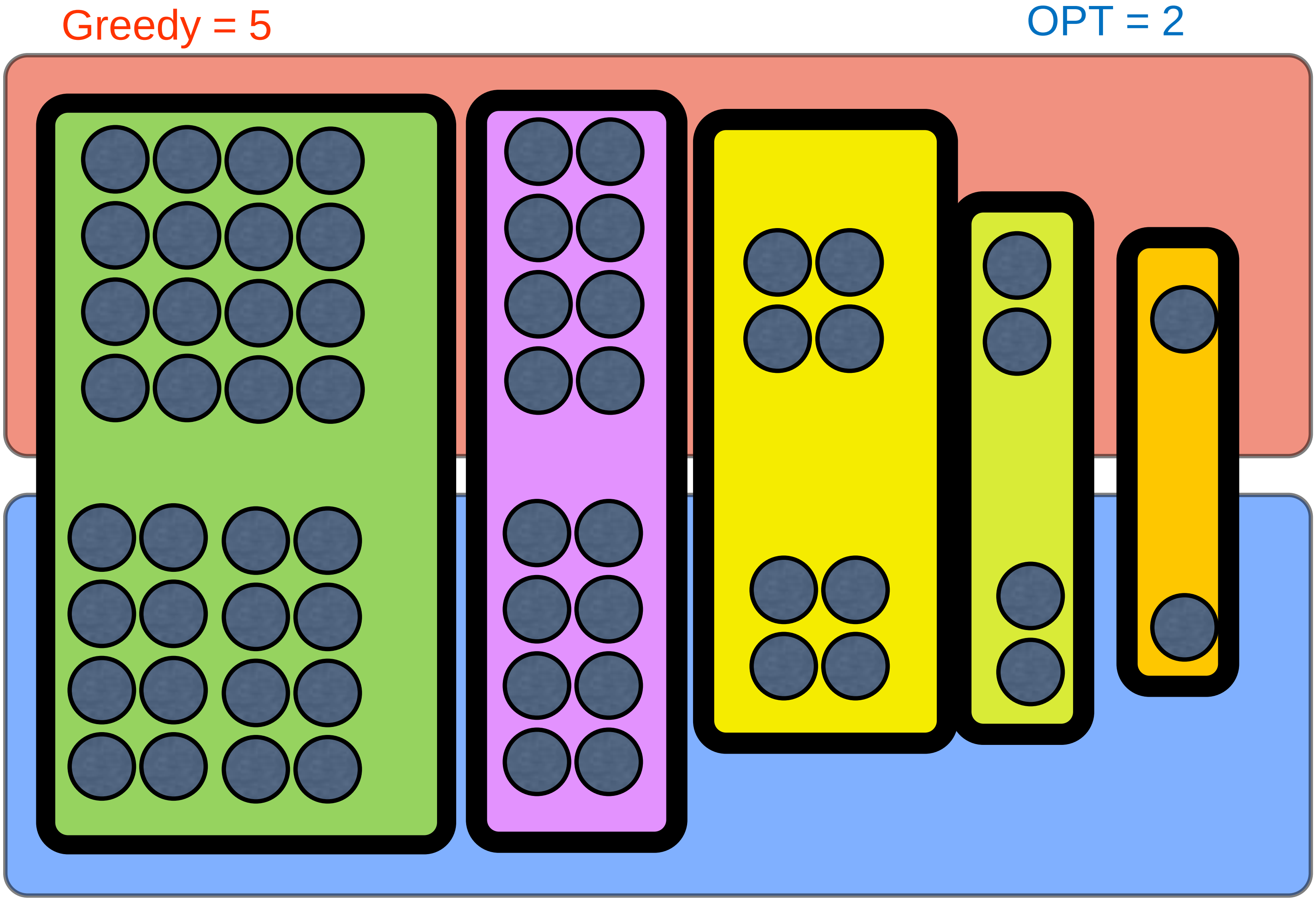
A Tight Example for Greedy



A Tight Example for Greedy



A Tight Example for Greedy



Greedy Gives O(log(n)) approximation

Thm: If the best solution has k sets, greedy finds at most k ln(n) sets.

Pf: Suppose OPT=k

There is set that covers 1/k fraction of remaining elements, since there are k sets that cover all remaining elements.

So in each step, algorithm will cover 1/k fraction of remaining elements.

#elements uncovered after t steps

$$\leq n\left(1-\frac{1}{k}\right)^{t} \leq e^{-t/k}$$

So after $t = k \ln n$ steps, # uncovered elements < 1.

Approximation Alg Summary

- To design approximation Alg, always find a way to lower bound OPT

- The best known approximation Alg for vertex cover is the greedy.
  - It has been open for 40 years to obtain a polynomial time algorithm with approximation ratio better than 2

- The best known approximation Alg for set cover is the greedy.
  - It is NP-Complete to obtain better than ln n approximation ratio for set cover.

Dynamic Programming

Algorithmic Paradigm

Greedy: Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer: Break up a problem into two sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems. Memorize the answers to obtain polynomial time ALG.

Dynamic Programming History

Bellman. Pioneered the systematic study of dynamic programming in the 1950s.

Etymology.

Dynamic programming = planning over time.

Secretary of Defense was hostile to mathematical research.

Bellman sought an impressive name to avoid confrontation.

- "it's impossible to use dynamic in a pejorative sense"
- "something not even a Congressman could object to"

Dynamic Programming Applications

Areas:

- Bioinformatics
- Control Theory
- Information Theory
- Operations Research
- Computer Science: Theory, Graphics, AI, …

Some famous DP algorithms

- Viterbi for hidden Markov Model
- Unix diff for comparing two files.
- Smith-Waterman for sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

Dynamic Programming

- Give a solution of a problem using smaller (overlapping) sub-problems where

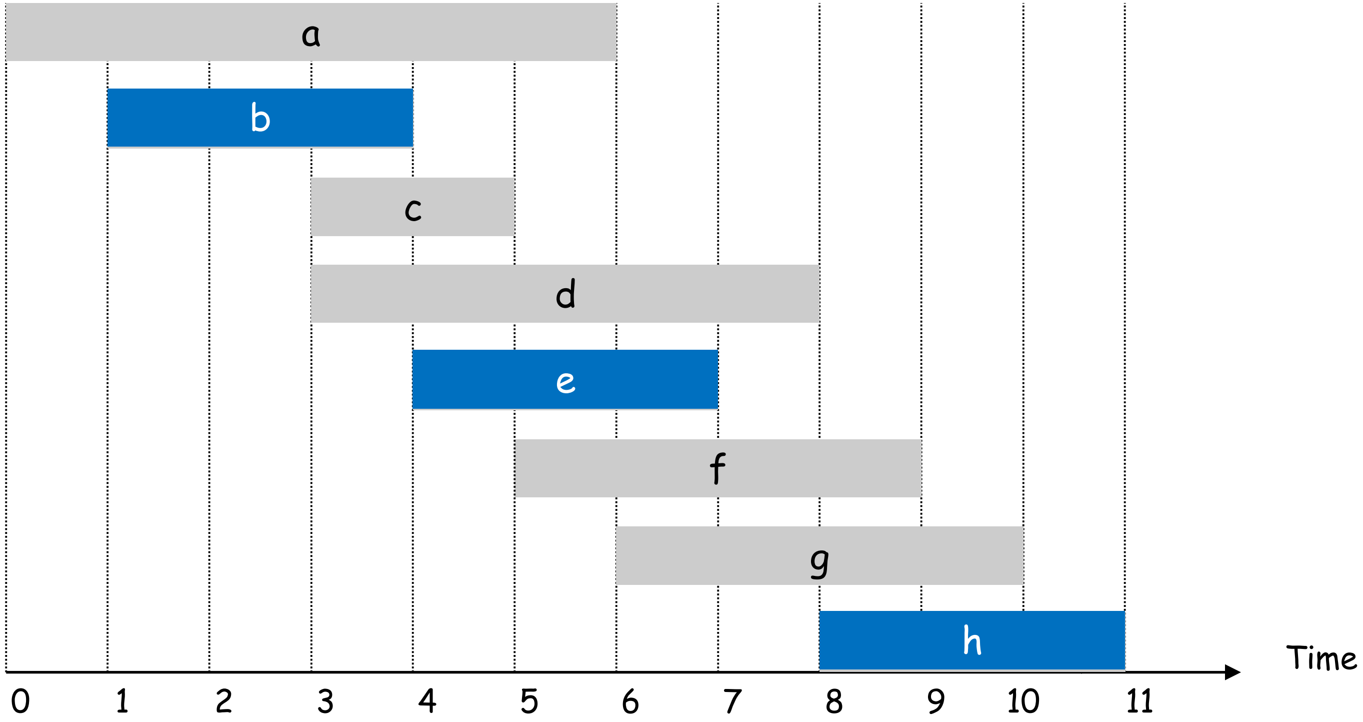
  the parameters of all sub-problems are determined in-advance

- Useful when the same subproblems show up again and again in the solution.

Weighted Interval Scheduling

Interval Scheduling

- Job j starts at $s(j)$ and finishes at $f(j)$ and has weight w_j
- Two jobs compatible if they don't overlap.
- Goal: find maximum weight subset of mutually compatible jobs.

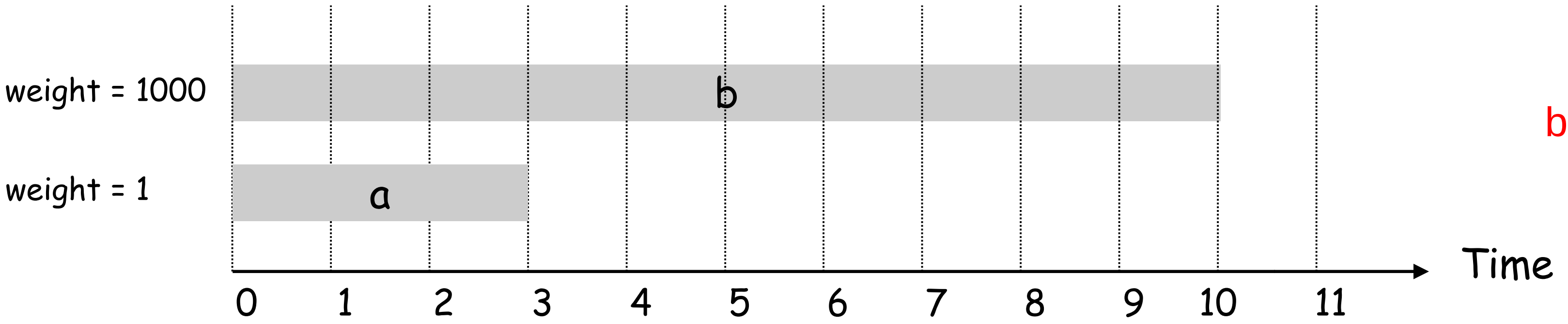


Unweighted Interval Scheduling: Review

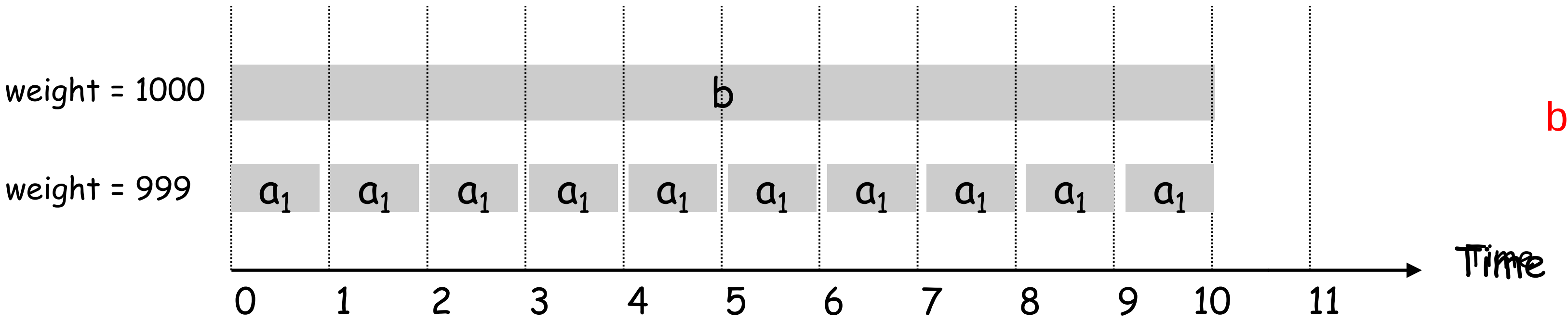
Recall: Greedy algorithm works if all weights are 1:

- Consider jobs in ascending order of finishing time
- Add job to a subset if it is compatible with prev added jobs.

OBS: Greedy ALG fails spectacularly (no approximation ratio) if arbitrary weights are allowed:



by finish



by weight

Weighted Job Scheduling by Induction

Suppose $1, \ldots, n$ are all jobs. Let us use induction:

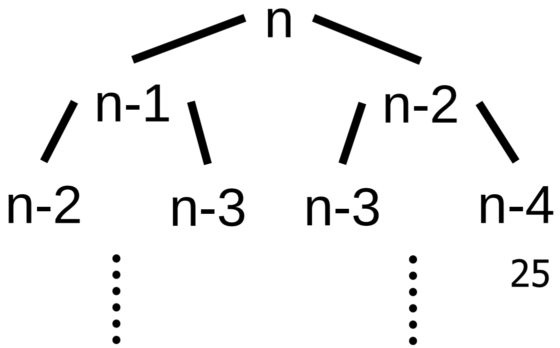
IH (strong ind): Suppose we can compute the optimum job scheduling for $< n$ jobs.

IS: Goal: For any n jobs we can compute OPT.
Case 1: Job n is not in OPT.
-- Then, just return OPT of $1, \ldots, n-1$.

Take best of the two

Case 2: Job n is in OPT.
-- Then, delete all jobs not compatible with n and recurse.

Q: Are we done?
A: No, How many subproblems are there?
Potentially 2^n all possible subsets of jobs.



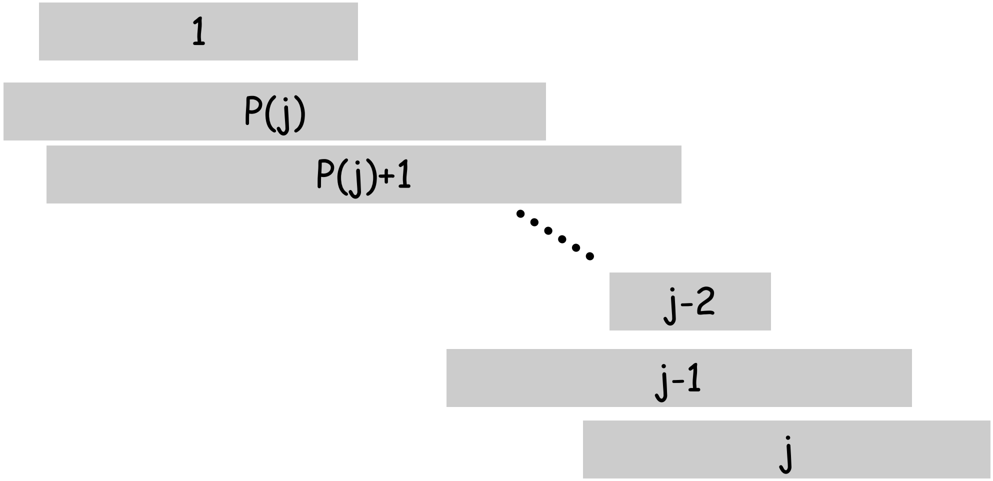
Sorting to Reduce Subproblems

IS: For jobs 1,…,n we want to compute OPT

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$

Case 1: Suppose OPT has job n.

- So, all jobs i that are not compatible with n are not OPT
- Let p(n) = largest index i < n such that job i is compatible with n.
- Then, we just need to find OPT of $1, \ldots, p(n)$



Sorting to reduce Subproblems

IS: For jobs 1,…,n we want to compute OPT

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$

Case 1: Suppose OPT has job n.

- So, all jobs i that are not compatible with n are not OPT
- Let p(n) = largest index i < n such that job i is compatible with n.
- Then, we just need to find OPT of $1, \ldots, p(n)$

Take best of the two

Case 2: OPT does not select job n.

- Then, OPT is just the optimum $1, \ldots, n-1$

Q: Have we made any progress (still reducing to two subproblems)?
A: Yes! This time every subproblem is of the form $1, \ldots, i$ for some i
So, at most n possible subproblems.

Weighted Job Scheduling by Induction

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$
Let OPT(j) denote the OPT solution of $1, \dots, j$

To solve OPT(j):
Case 1: OPT(j) has job j.

- So, all jobs i that are not compatible with j are not OPT(j)
- Let p(j) = largest index i < j such that job i is compatible with j.
- So $OPT(j) = OPT\big(p(j)\big) \cup \{j\}$.

Case 2: OPT(j) does not select job j.

- Then, $OPT(j) = OPT(j-1)$

$$\boxed{OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\big(w_j + OPT\big(p(j)\big), OPT(j-1)\big) & \text{o.w.} \end{cases}}$$

Algorithm

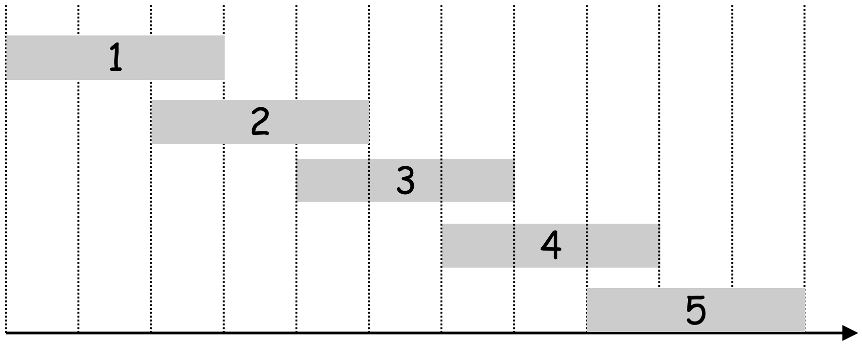
```
Input: n, $s(1),\dots,s(n)$  and $f(1),\dots,f(n)$ and $w_1,\dots,w_n$.

Sort jobs by finish times so that $f(1) \le f(2) \le \cdots f(n)$.

Compute $p(1), p(2), \dots, p(n)$

Compute-Opt(j) {
   if (j = 0)
      return 0
   else
      return max($w_j$ + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

Recursive Algorithm Fails

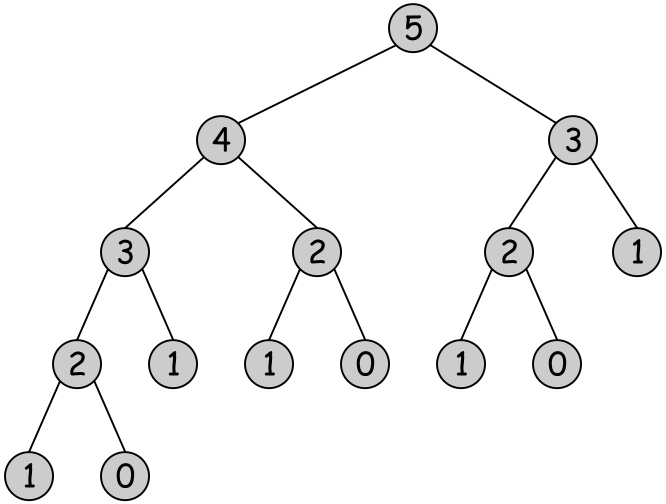
Even though we have only n subproblems, we do not store the solution to the subproblems

- So, we may re-solve the same problem many many times.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence



p(1) = 0, p(j) = j-2



Algorithm with Memorization

Memoization. Compute and Store the solution of each sub-problem in a cache the first time that you face it. lookup as needed.

```
Input: n, $s(1),\ldots,s(n)$  and $f(1),\ldots,f(n)$ and $w_1,\ldots,w_n$.

Sort jobs by finish times so that $f(1) \le f(2) \le \cdots f(n)$.

Compute $p(1),p(2),\ldots,p(n)$

for j = 1 to n
   M[j] = empty
M[0] = 0

M-Compute-Opt(j) {
   if (M[j] is empty)
      M[j] = max($w_j$ + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]
}
```

Bottom up Dynamic Programming

You can also avoid recusion

- recursion may be easier conceptually when you use induction

```
Input: n, $s(1), \ldots, s(n)$  and $f(1), \ldots, f(n)$ and $w_1, \ldots, w_n$.

Sort jobs by finish times so that $f(1) \leq f(2) \leq \cdots f(n)$.

Compute $p(1), p(2), \ldots, p(n)$

Iterative-Compute-Opt {
   M[0] = 0
   for j = 1 to n
      M[j] = max($w_j$ + M[p(j)], M[j-1])
}

Output M[n]
```

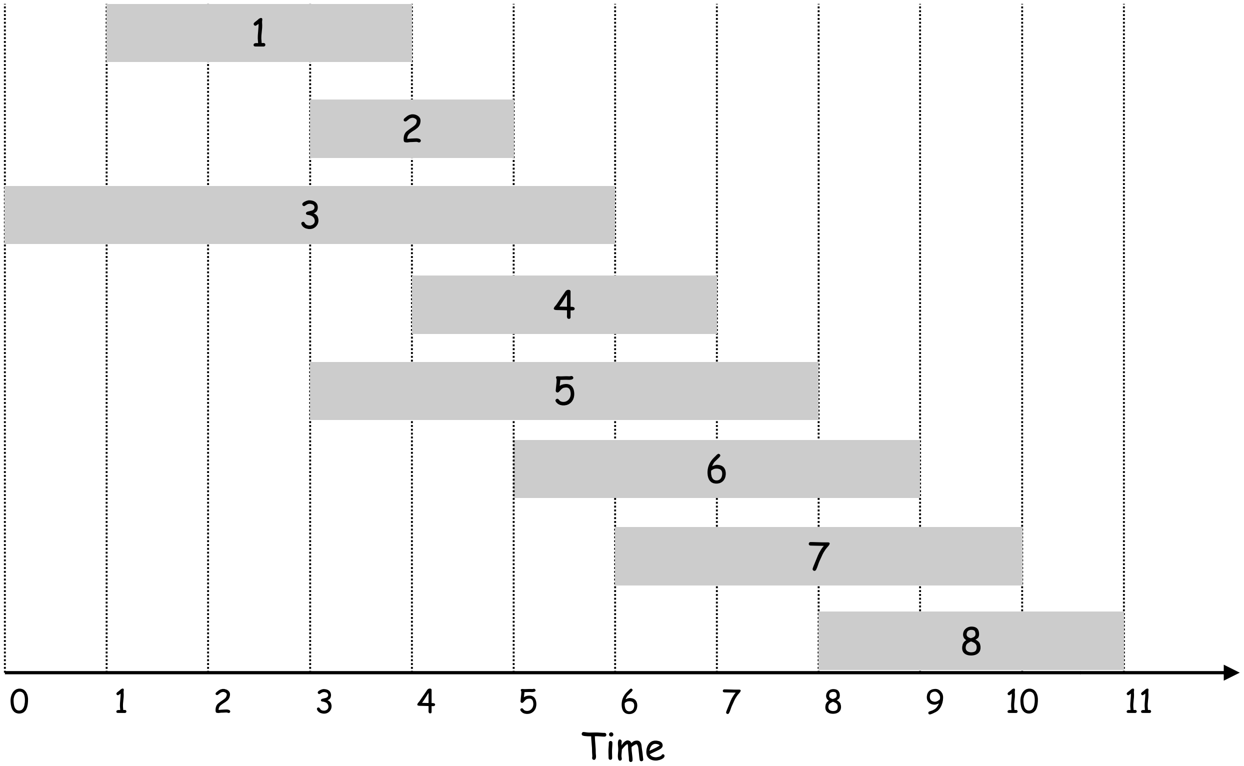
Claim: M[j] is value of OPT(j)

Timing: Easy. Main loop is O(n); sorting is O(n log n)

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

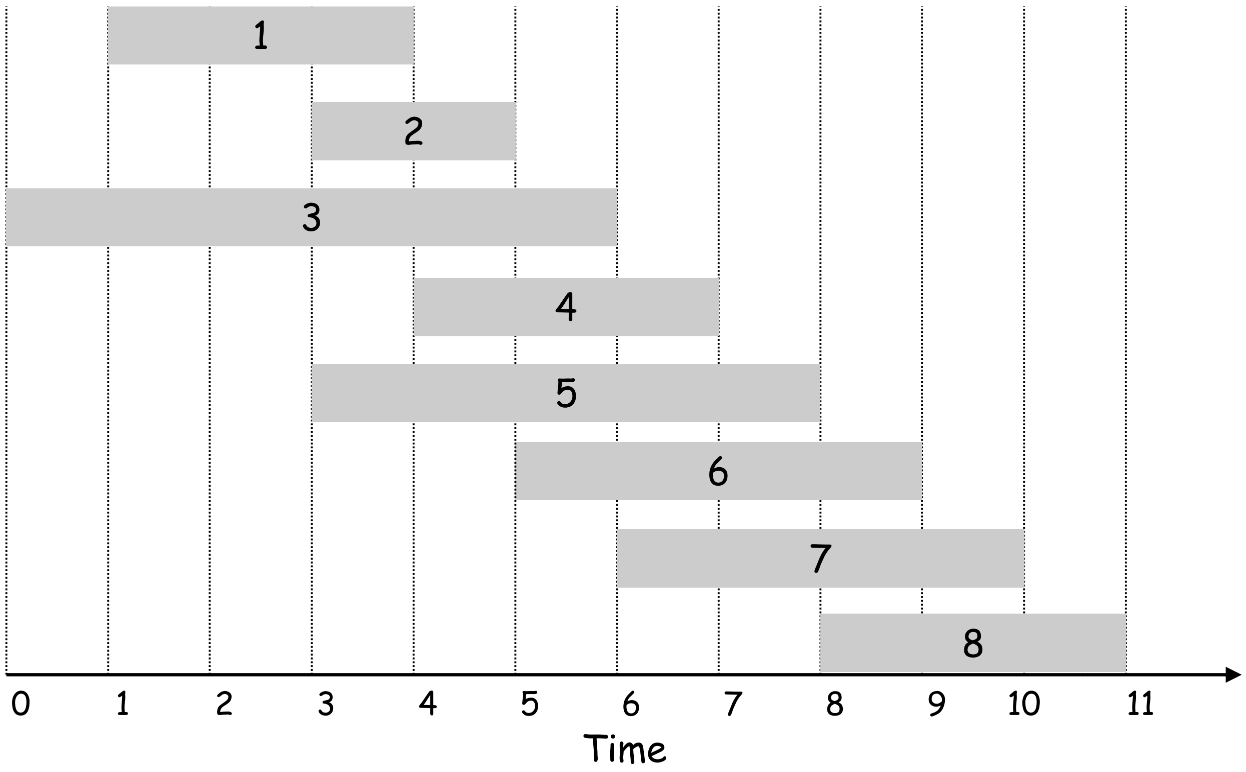


j	w_j	p(j)	OPT(j)
0			0
1	3	0	
2	4	0	
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

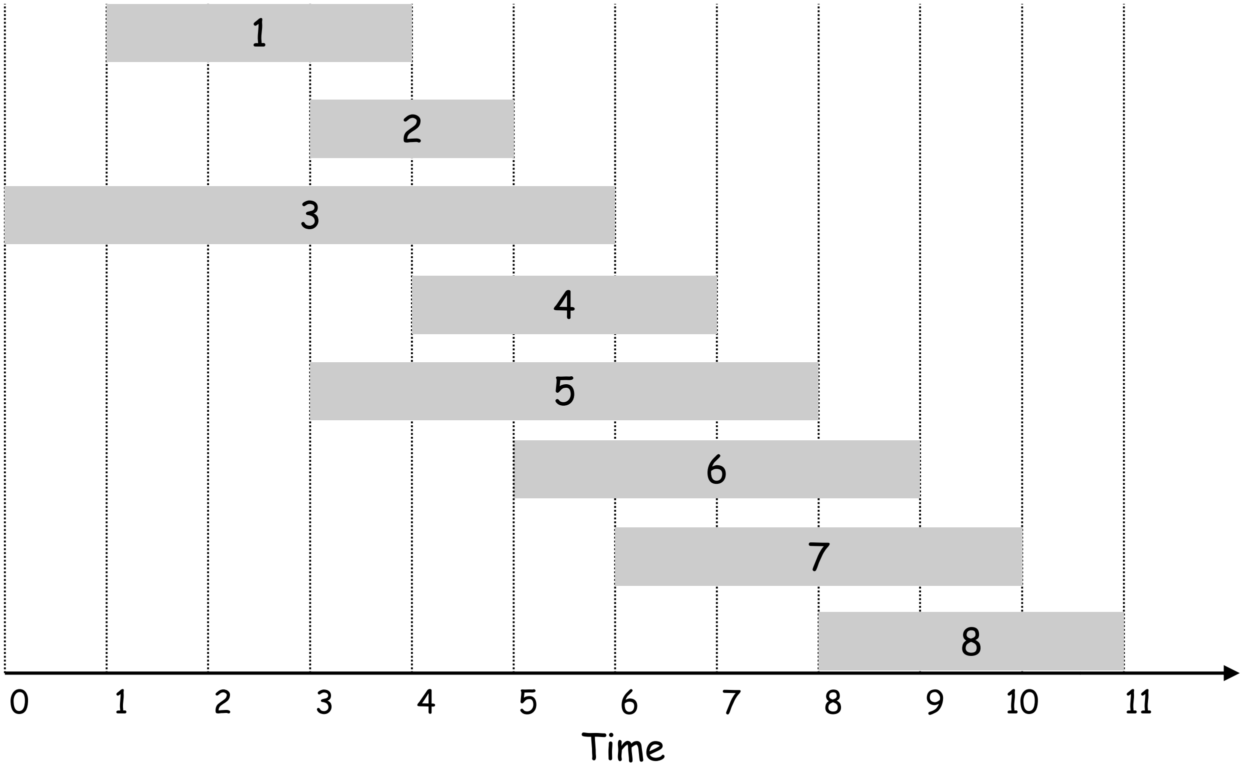


j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

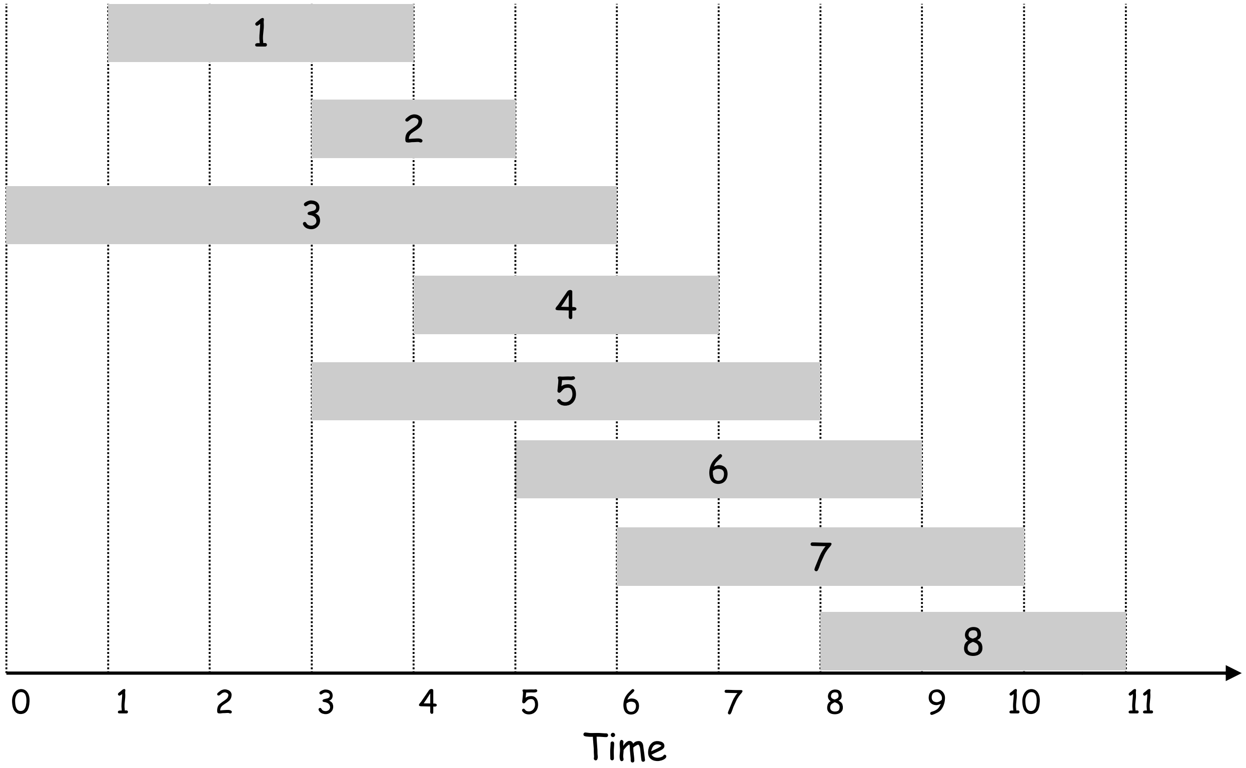


j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	**4**
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

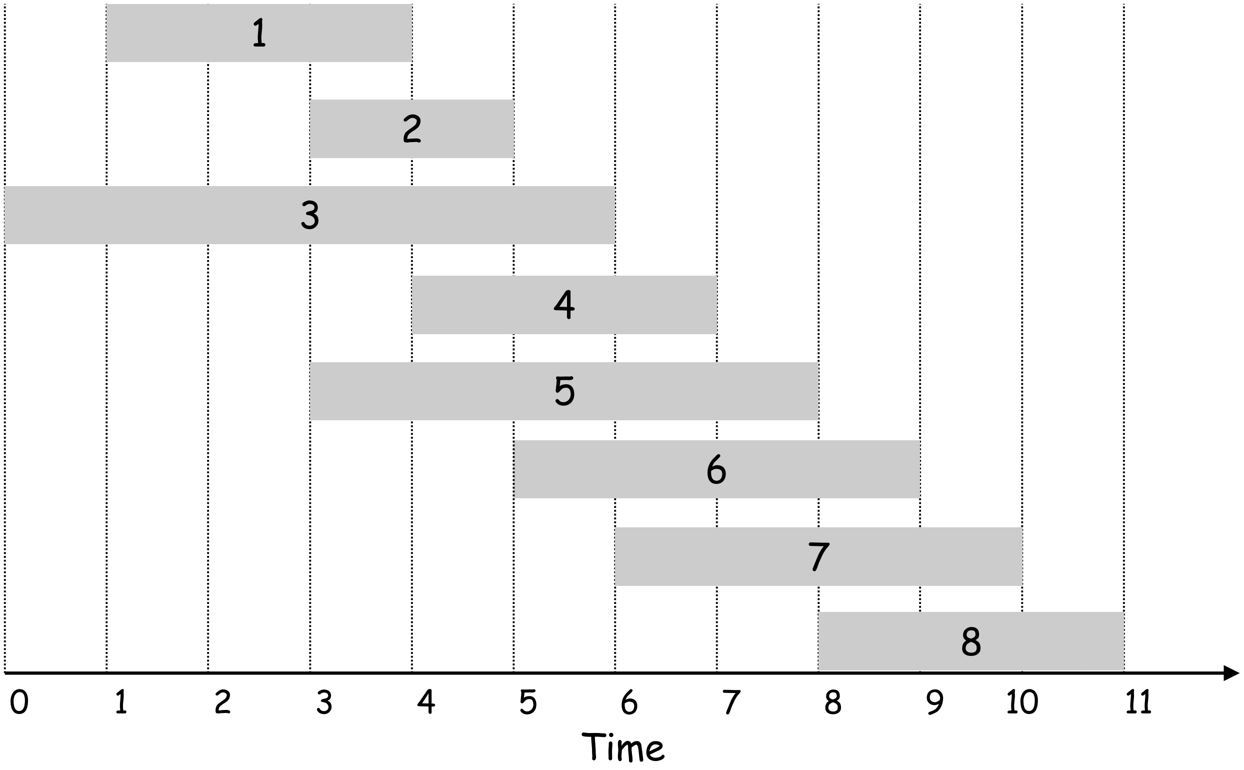


j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	**4**
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

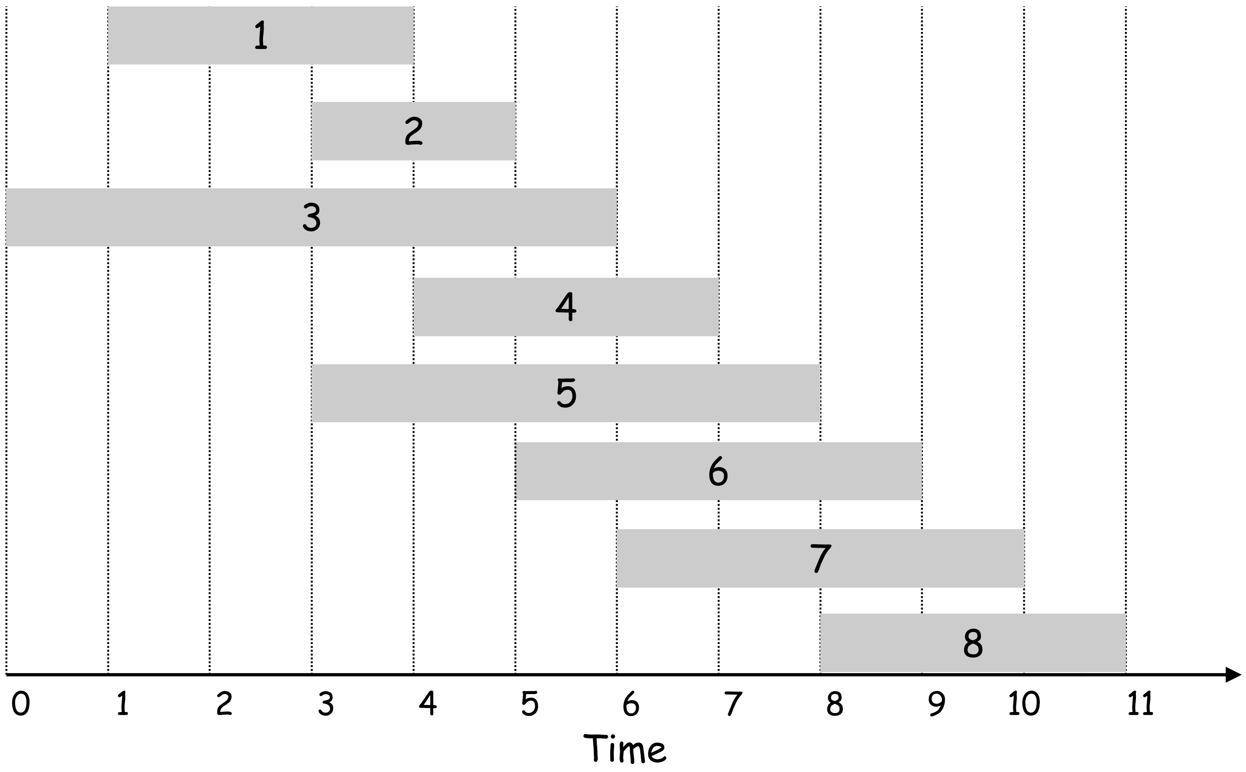


j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	**6**
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

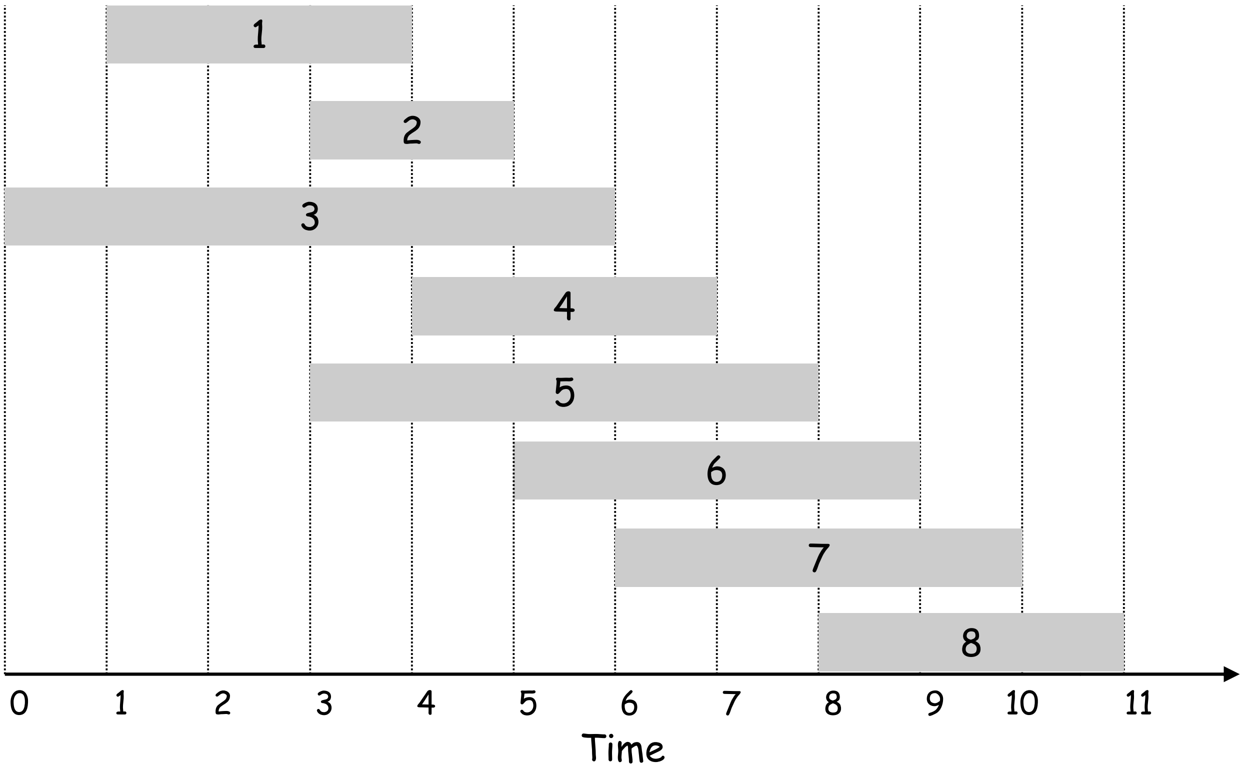


j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \le \cdots \le f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

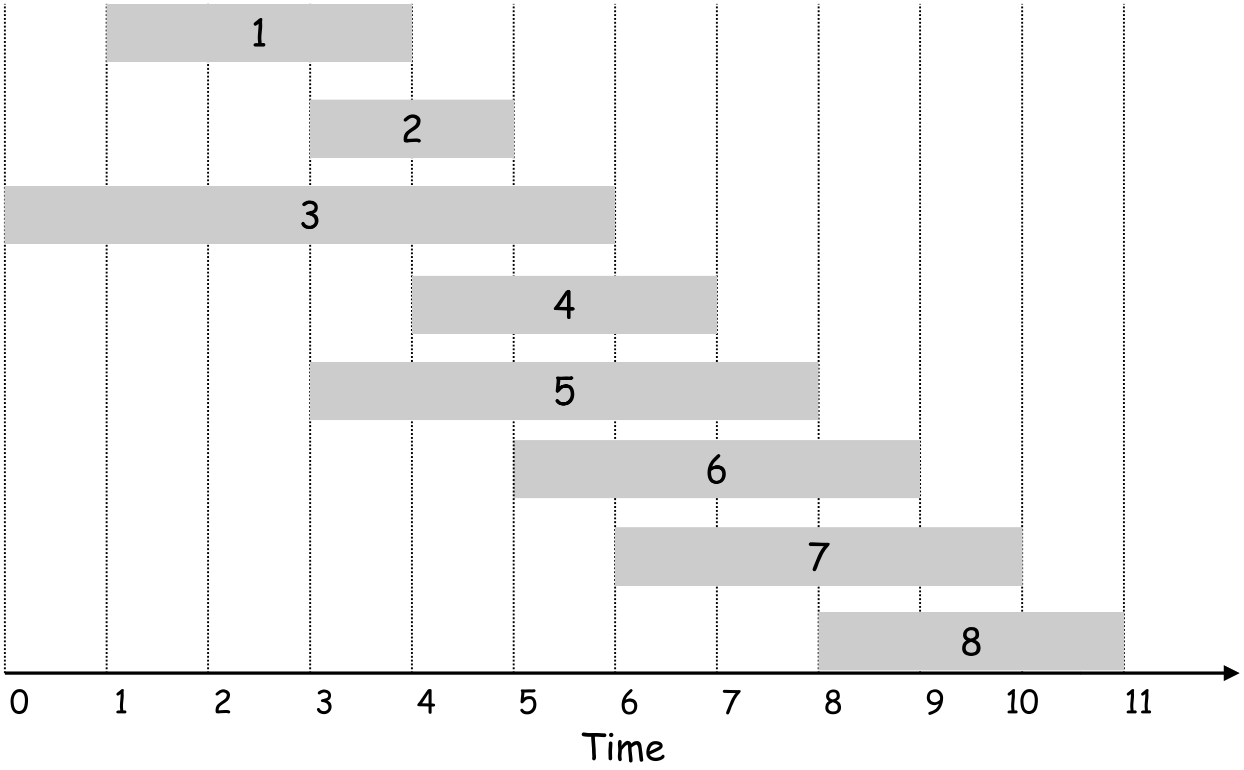


j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

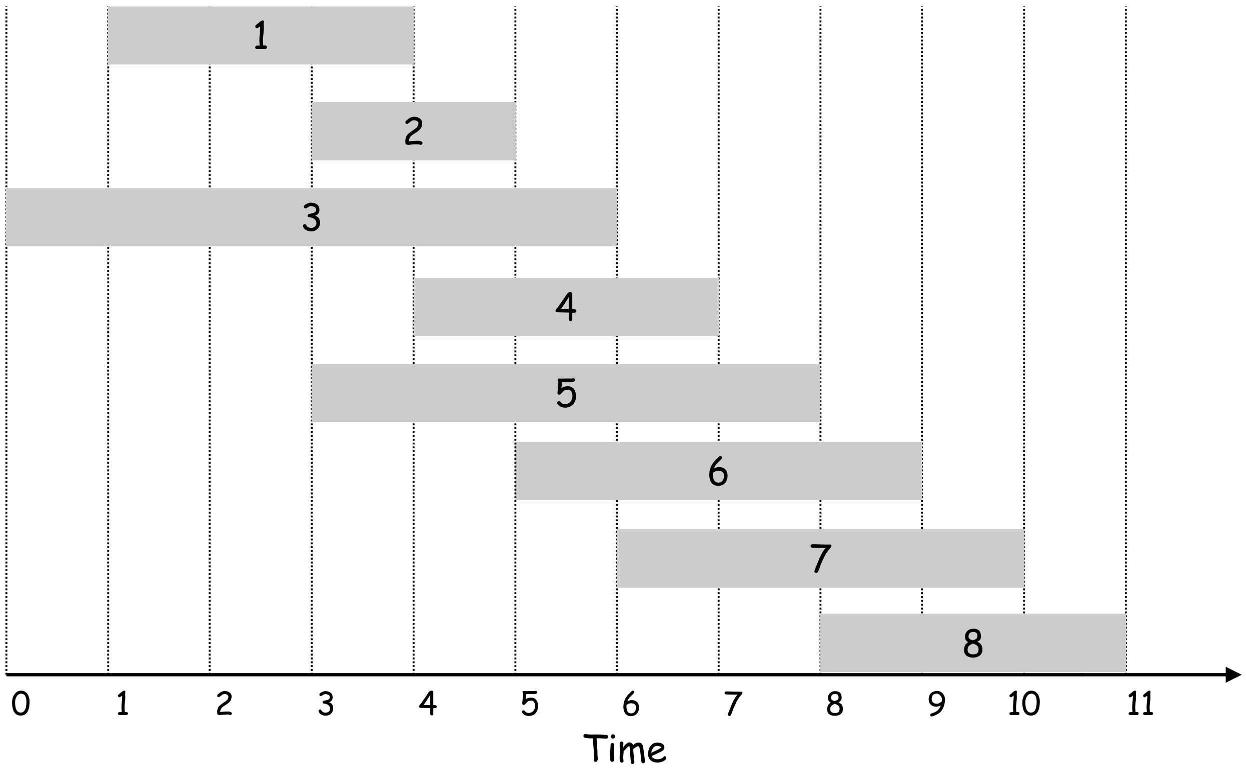


j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	**7**
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

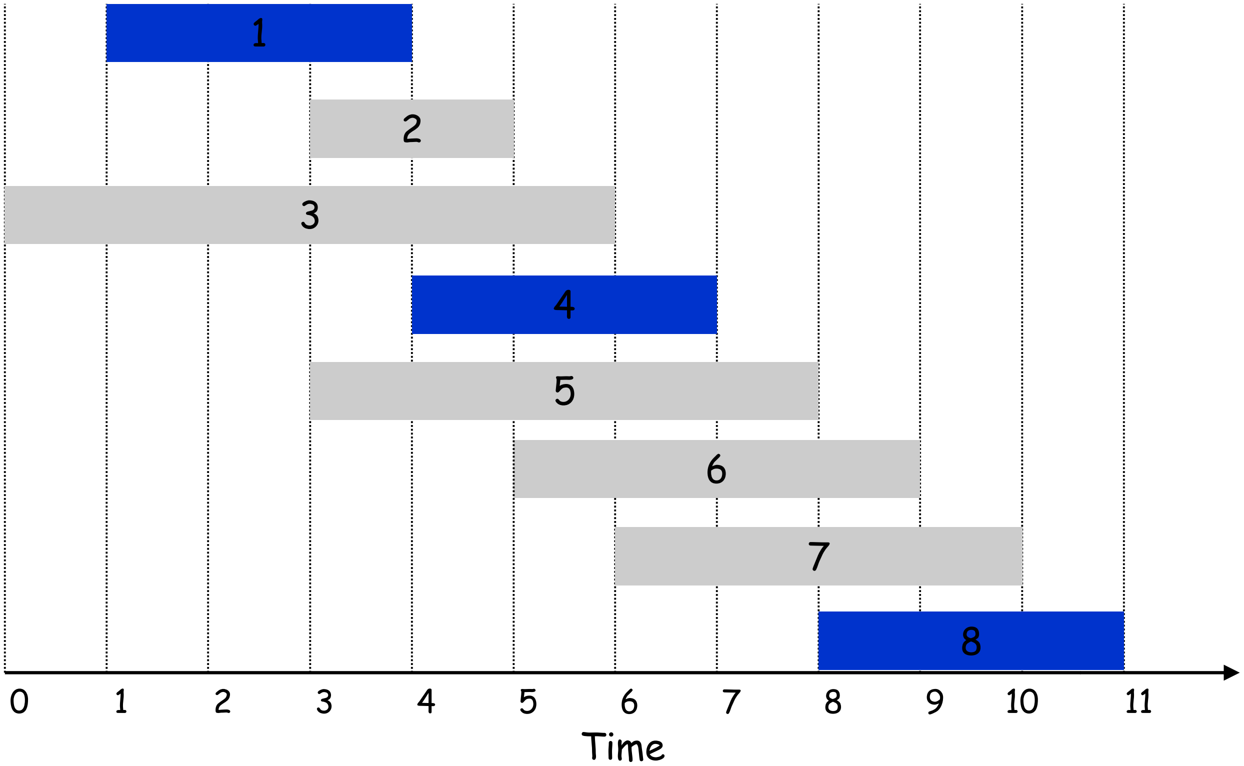


j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	10

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.



j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	**10**

Knapsack Problem

Knapsack Problem

Given n objects and a "knapsack.“

Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.

Knapsack has capacity of W kilograms.

Goal: fill knapsack so as to maximize total value.

Ex: OPT is { 3, 4 } with value 40.

$\boxed{W = 11}$

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i.

Ex: { 5, 2, 1 } achieves only value = 35 $\Rightarrow$ greedy not optimal.