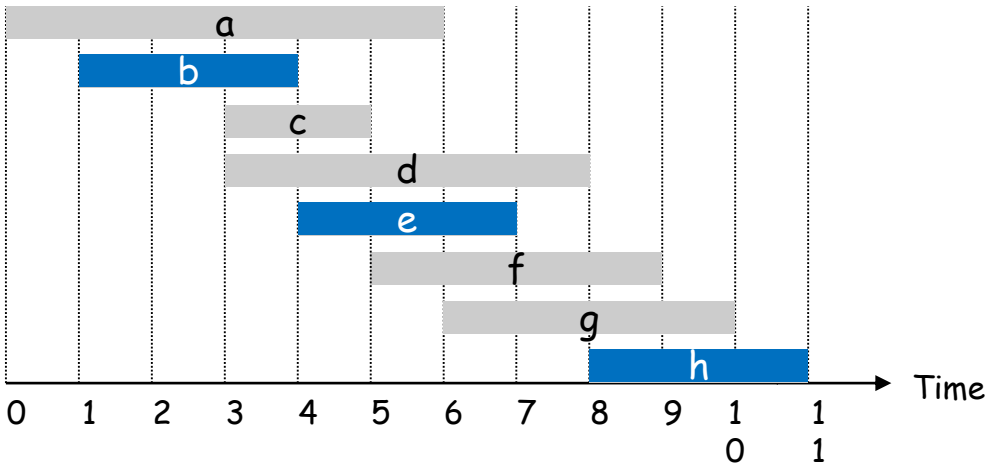
CSE 421

Greedy Alg: Interval Partitioning / Job Scheduling

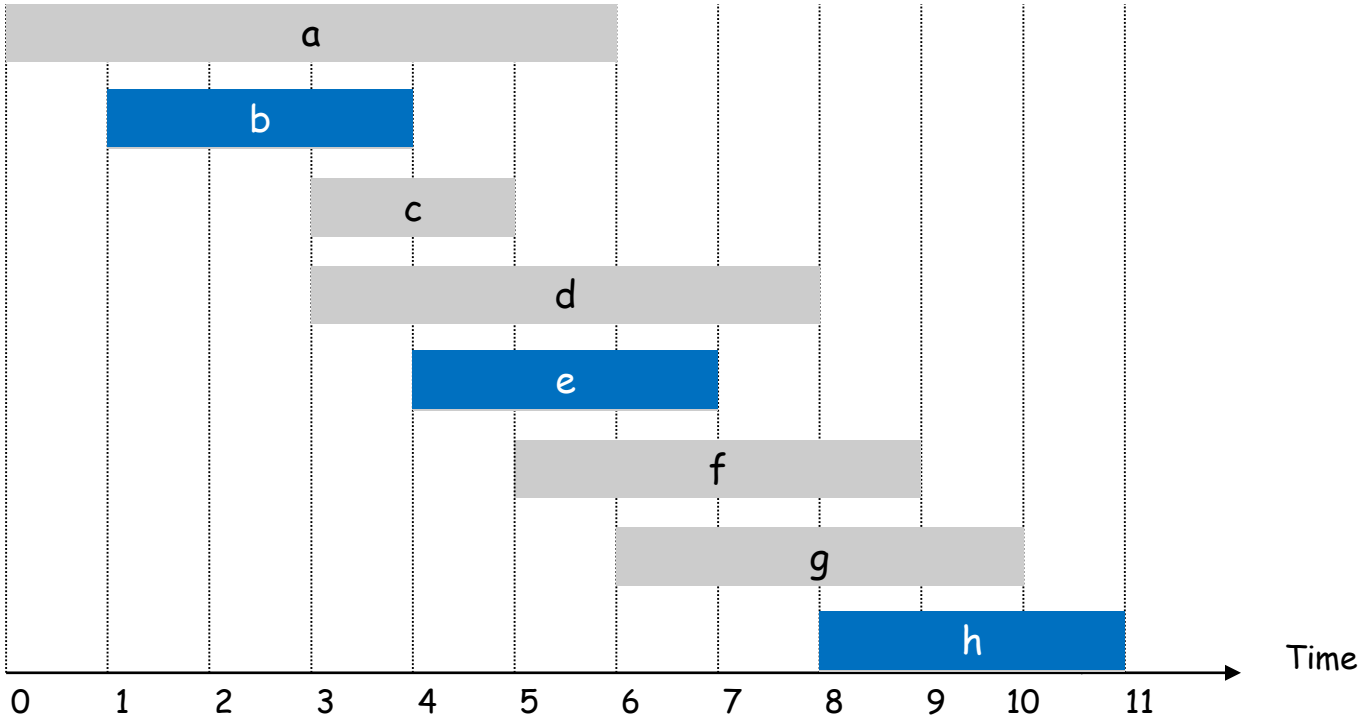
Shayan Oveis Gharan

Interval Scheduling



Interval Scheduling

- Job j starts at $s(j)$ and finishes at $f(j)$.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Greedy Alg: Earliest Finish Time

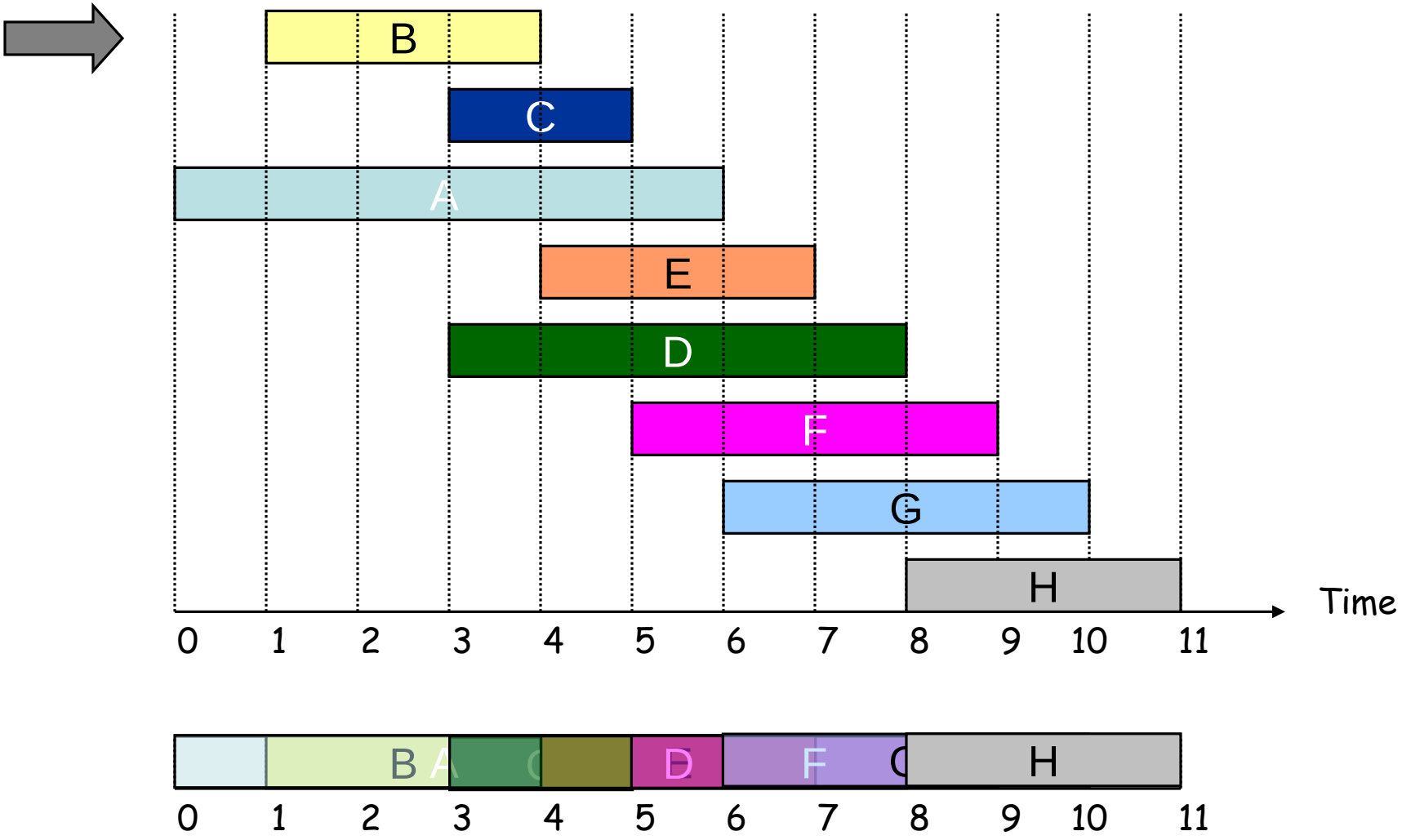
Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that f(1) ≤ f(2) ≤ ... ≤ f(n).
A ← ∅
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

Implementation. O(n log n).

- Remember job j* that was added last to A.
- Job j is compatible with A if $s(j) \geq f(j^*)$.

Greedy Alg: Example



Correctness

Theorem: Greedy algorithm is optimal.

Pf: (technique: "Greedy stays ahead")

Let $i_1, i_2, \ldots i_k$ be jobs picked by greedy, $j_1, j_2, \ldots j_m$ those in some optimal solution in order.

We show $f(i_r) \leq f(j_r)$ for all r, by induction on r.

Base Case: i_1 chosen to have min finish time, so $f(i_1) \leq f(j_1)$.

IH: $f(i_r) \leq f(j_r)$ for some r

IS: Since $f(i_r) \leq f(j_r) \leq s(j_{r+1})$, j_{r+1} is among the candidates considered by greedy when it picked i_{r+1}, & it picks min finish, so $f(i_{r+1}) \leq f(j_{r+1})$

Observe that we must have $k \geq m$, else j_{k+1} is among (nonempty) set of candidates for i_{k+1}

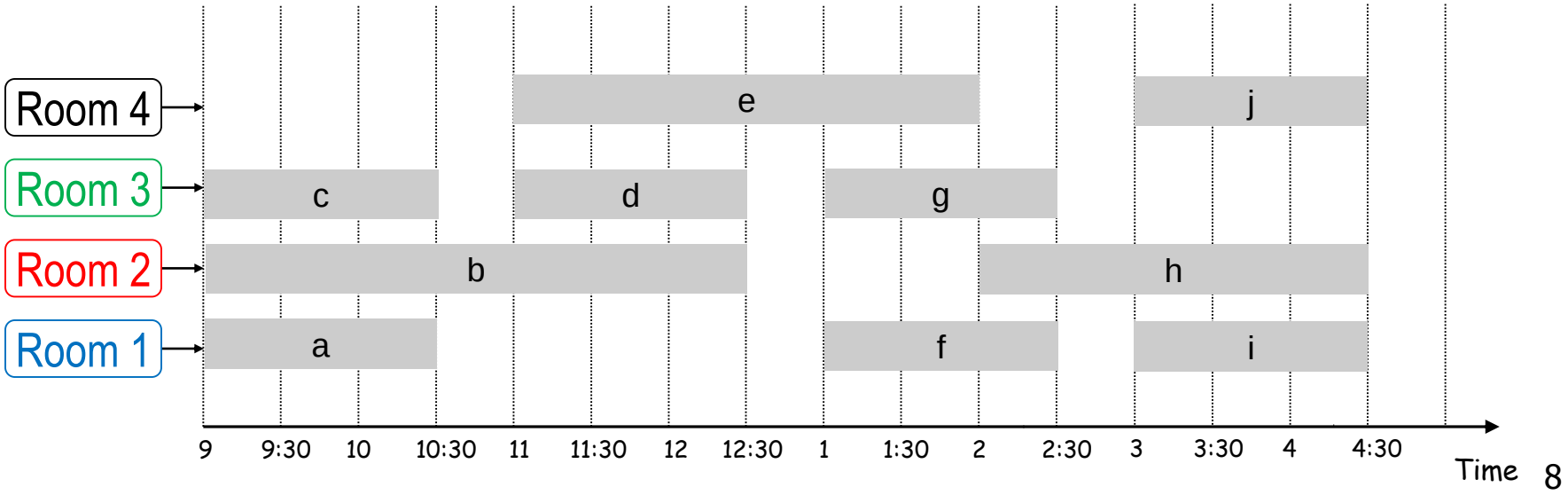
■

Interval Partitioning Technique: Structural

Interval Partitioning

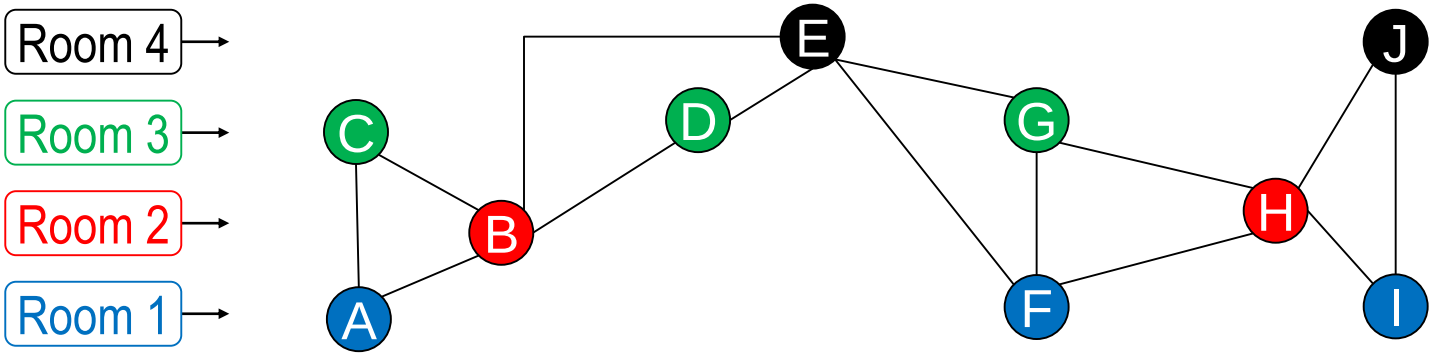
Lecture j starts at s(j) and finishes at f(j).

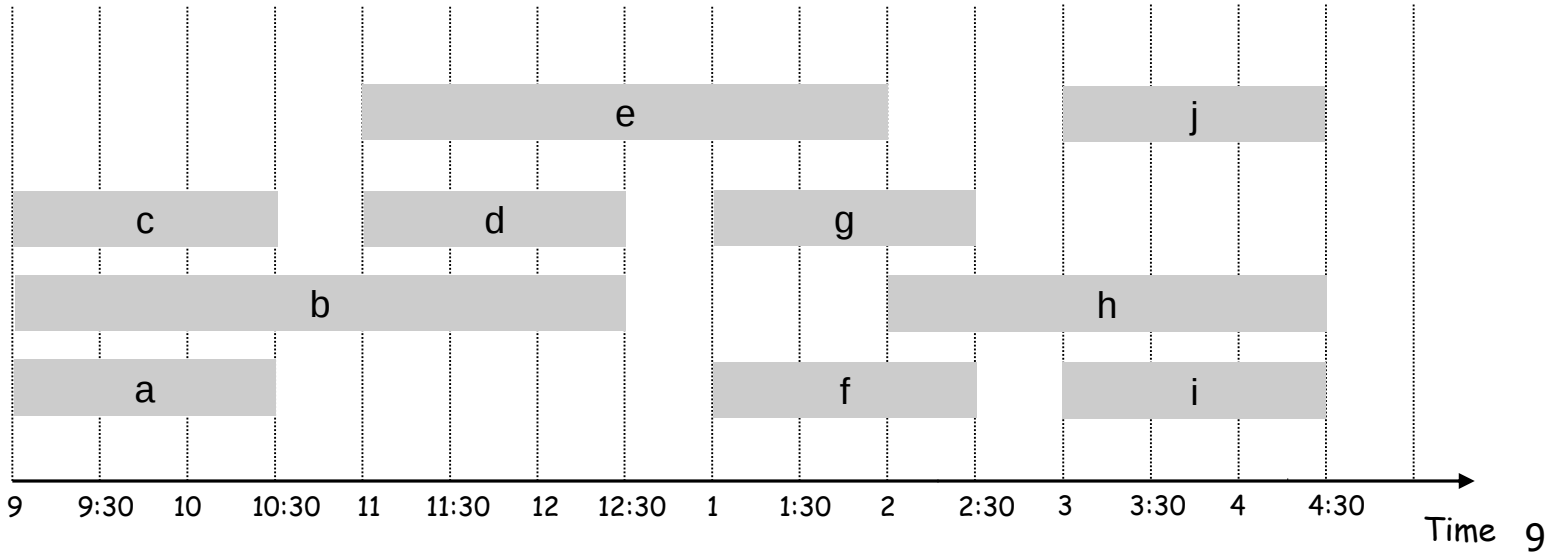
Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.



Interval Partitioning

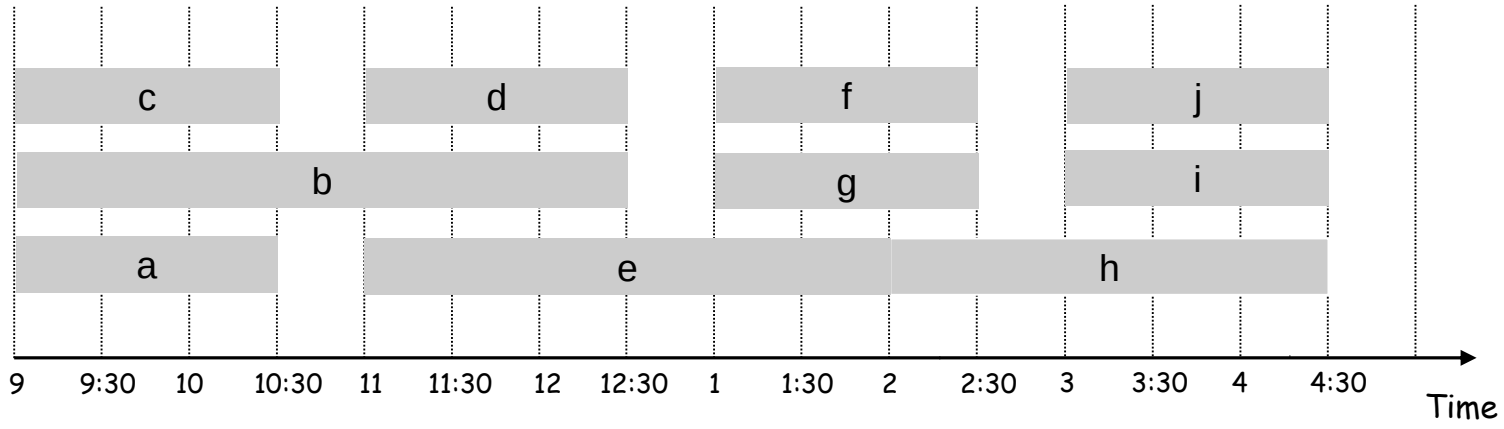
Note: graph coloring is very hard in general, but graphs corresponding to interval intersections are simpler.





A Better Schedule

This one uses only 3 classrooms



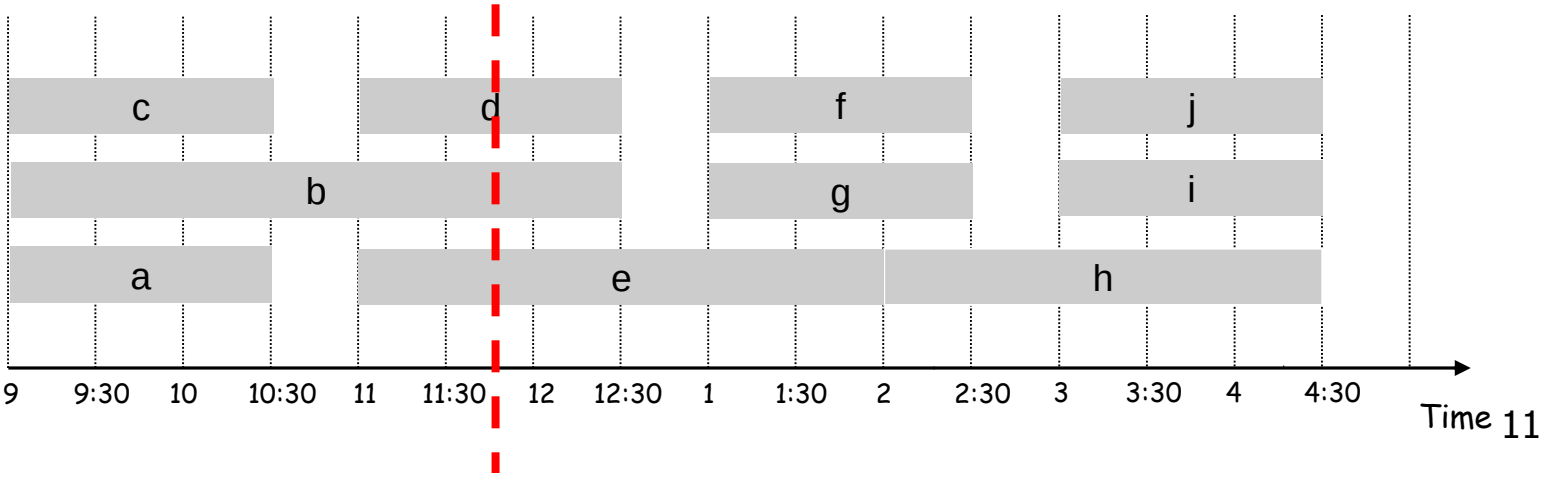
A Structural Lower-Bound on OPT

Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of schedule below = 3 $\Rightarrow$ schedule below is optimal.

Q. Does there always exist a schedule equal to depth of intervals?



A Greedy Algorithm

Greedy algorithm: Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s_1 ≤ s_2 ≤ ... ≤ s_n.
d ← 0

for j = 1 to n {
   if (lect j is compatible with some classroom k, 1 ≤ k ≤ d)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation: Exercise!

Correctness

Observation: Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem: Greedy algorithm is optimal.

Pf (exploit structural property).

Let d = number of classrooms that the greedy algorithm allocates.

Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 previously used classrooms.

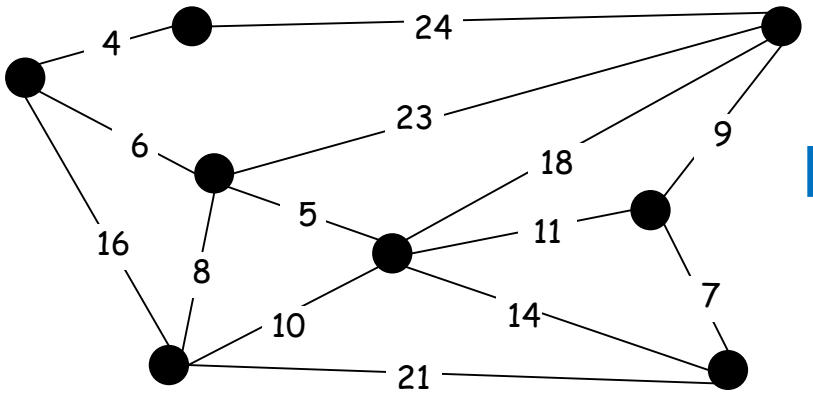
Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s(j).

Thus, we have d lectures overlapping at time $s(j) + \epsilon$, i.e. depth $\geq$ d

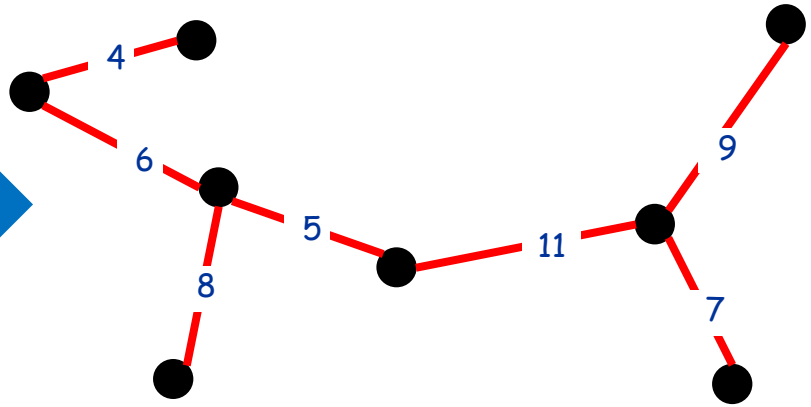
"OPT Observation" $\Rightarrow$ all schedules use $\geq$ depth classrooms, so d = depth and greedy is optimal ▪

Minimum Spanning Tree (MST)

Given a connected graph $G = (V, E)$ with real-valued edge weights c_e, an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge weights is minimized.



$G = (V, E)$



$T, \sum_{e \in T} c_e = 50$

Applications

Network design:

- telephone, electrical, hydraulic, TV cable, computer, road

Approximation algorithms for NP-hard problems:

- traveling salesperson problem, Steiner tree

Indirect applications:

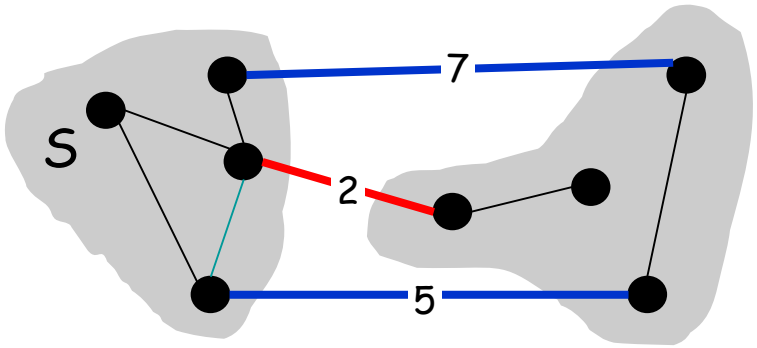
- Graph clustering
- max bottleneck paths
- LDPC codes for error correction
- image registration with Renyi entropy
- learning salient features for real-time face verification
- reducing data storage in sequencing amino acids in a protein
- model locality of particle interactions in turbulent fluid flows
- autoconfig protocol for Ethernet bridging to avoid cycles in a network

Properties of the OPT

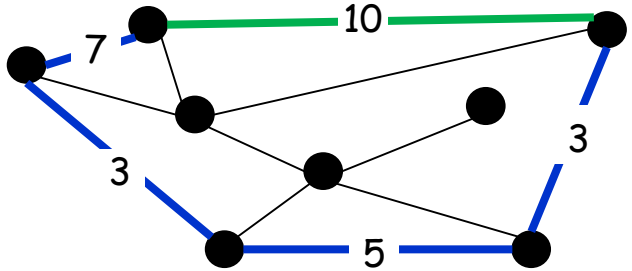
Simplifying assumption: All edge costs c_e are distinct.

Cut property: Let S be any subset of nodes (called a cut), and let e be the min cost edge with exactly one endpoint in S. Then every MST contains e.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then no MST contains f.



red edge is in the MST

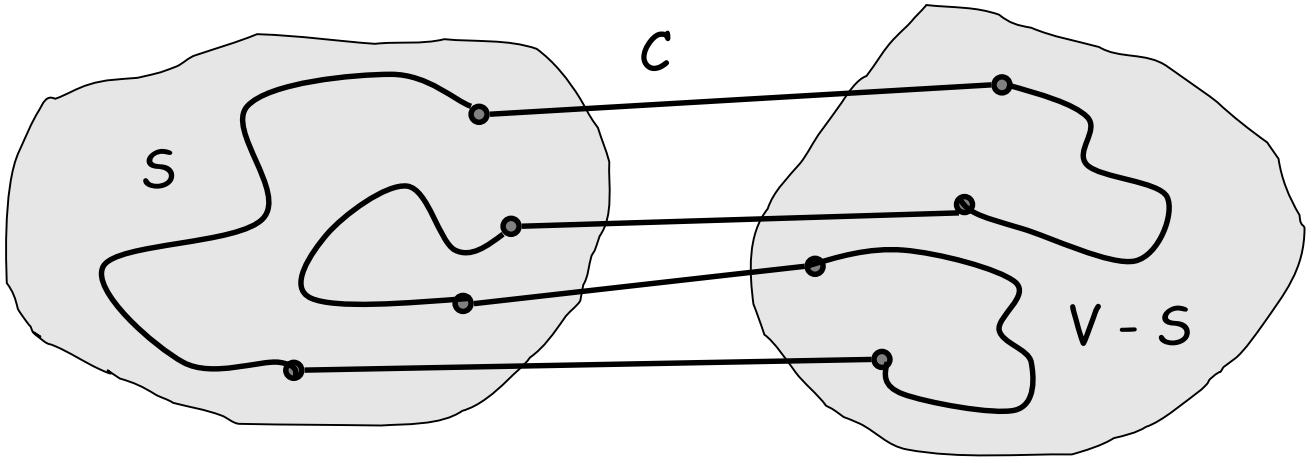


Green edge is not in the MST

Cycles and Cuts

Claim. A cycle crosses a cut (from S to V-S) an even number of times.

Pf. (by picture)



Cut Property: Proof

Simplifying assumption: All edge costs c_e are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the T* contains e.

Pf. By contradiction

Suppose e = {u,v} does not belong to T*.

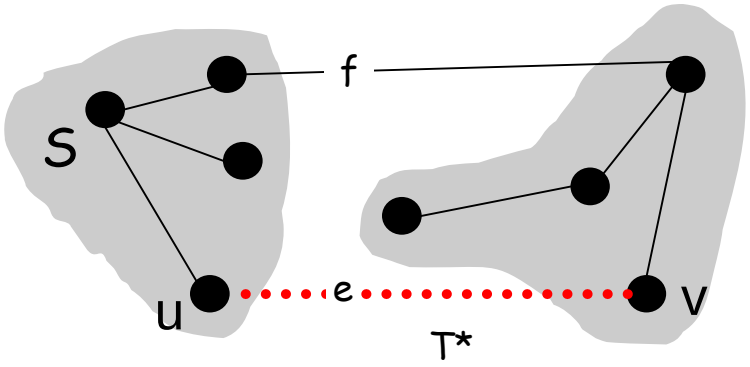
Adding e to T* creates a cycle C in T*.

There is a path from u to v in T* $\Rightarrow$ there exists another edge, say f, that leaves S.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $\text{cost}(T) < \text{cost}(T^*)$.

This is a contradiction.



Cycle Property: Proof

Simplifying assumption: All edge costs c_e are distinct.

Cycle property: Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

Pf. (By contradiction)

Suppose f belongs to T*.

Deleting f from T* cuts T* into two connected components.

There exists another edge, say e, that is in the cycle and connects the components.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $\text{cost}(T) < \text{cost}(T^*)$.

This is a contradiction.