| **CSE421: Design and Analysis of Algorithms** | January 4, 2018 |
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| **Homework 1** | |
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Please see https://courses.cs.washington.edu/courses/cse421/18wi/grading.html for general guidelines about Homework problems.

Most of the problems only require one or two key ideas for their solution. It will help you a lot to spell out these main ideas so that you can get most of the credit for a problem even if you err on the finer details. Please justify all answers.

P1) Let $m_1, m_2$ be two of the men and $w_1, w_2$ be two of the women in an instance of stable matching problem with $n$ men and $n$ women. Suppose that $m_1$'s preference list is $w_1 > w_2 > \ldots$ and $m_2$'s preference list is $w_2 > w_1 > \ldots$. In addition, suppose $w_1$'s preference list is $m_2 > m_1 > \ldots$ and $w_2$'s preference list is $m_1 > m_2 > \ldots$. So, we only know the favorite and the second favorite of each of these four persons.

    a) Show that in the output of the Gale-Shapley algorithm, $m_1$ is matched to $w_1$ and $m_2$ is matched to $w_2$.

    b) Show that in *every* stable matching, $m_1, m_2$ are matched to $w_1, w_2$, i.e., we either have $(w_1, m_1), (w_2, m_2)$ or $(w_1, m_2), (w_2, m_1)$ as part of the stable matching.

    c) **Mini Extra Credit:** Construct an instance of stable matching problem with $n$ men and $n$ women with $2^{n/2}$ stable matchings. Justify your claim.

P2) Suppose the preference lists of all men in the stable matching problem is equal to $1 > 2 > \cdots > n$. That is they all prefer the first women, then the second woman and so on. Show that the Gale-Shapley algorithm makes at least $n^2/2$ proposals.

P3) Gale and Shapley published their paper on the stable marriage problem in 1962; but a version of their algorithm had already been in use for ten years by the National Resident Matching Program, for the problem of assigning medical residents to hospitals.

Basically, the situation was the following. There were $m$ hospitals, each with a certain number of available positions for hiring residents. There were $n$ medical students graduating in a given year, each interested in joining one of the hospitals. Each hospital had a ranking of the students in order of preference, and each student had a ranking of the hospitals in order of preference. We will assume that there were more students graduating than there were slots available in the $m$ hospitals.

The interest, naturally, was in finding a way of assigning each student to at most one hospital, in such a way that all available positions in all hospitals were filled. (Since we are assuming a surplus of students, there would be some students who do not get assigned to any hospital.) We say that an assignment of students to hospitals is stable if neither of the following situations arises.

    • **First type of instability:** There are students $s$ and $s'$, and a hospital $h$, so that $s$ is assigned to $h$, and $s'$ is assigned to no hospital, and $h$ prefers $s'$ to $s$.

- **Second type of instability:** There are students $s$ and $s'$, and hospitals $h$ and $h'$, so that

  - $s$ is assigned to $h$, and
  - $s'$ is assigned to $h'$, and
  - $h$ prefers $s'$ to $s$, and $s'$ prefers $h$ to $h'$.

So we basically have the stable marriage problem from class, except that (i) hospitals generally want more than one resident, and (ii) there is a surplus of medical students.

Show that there is always a stable assignment of students to hospitals, and give an efficient algorithm to find one.

P4) **Extra Credit:** Design an algorithm that outputs all stable matchings of a given instance with $n$ men and $n$ women such that your algorithm runs in time polynomial in $n$ and the number of stable matchings of the input instance.