CSE 421

Edmonds-Karp Algorithm, More Applications

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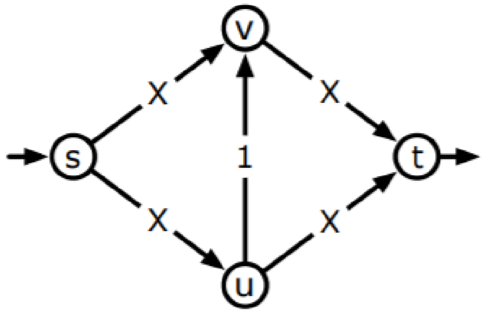
FF may converge to wrong answer

- FF is not polynomial time.

It can take X steps for this graph.



- FF may not even converge to a correct flow for irrational capacity.

$\phi = \frac{\sqrt{5}-1}{2}$, $X = 3$.

Augmenting paths along

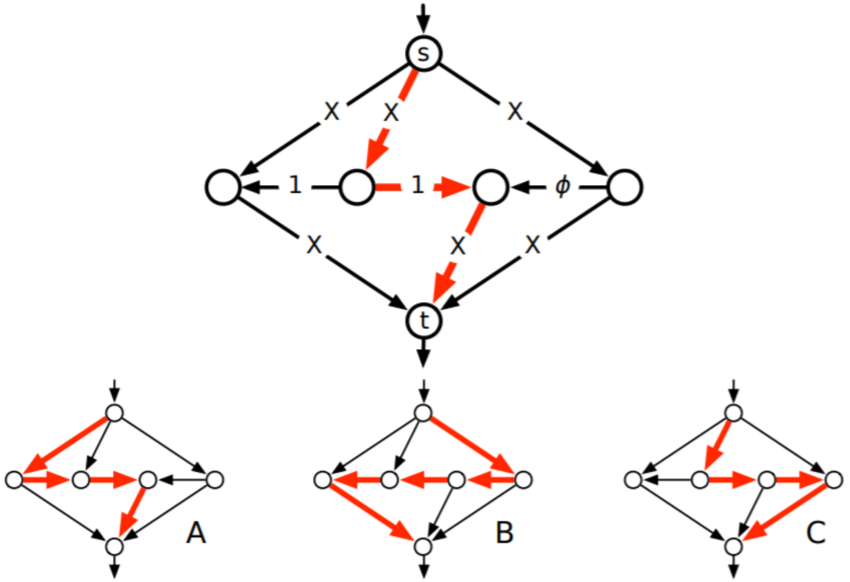
the large figure.

Then,

B,C,B,A,B,C,B,A,B,C,B,A,…

In limit, it will send $4 + \sqrt{5}$ unit.

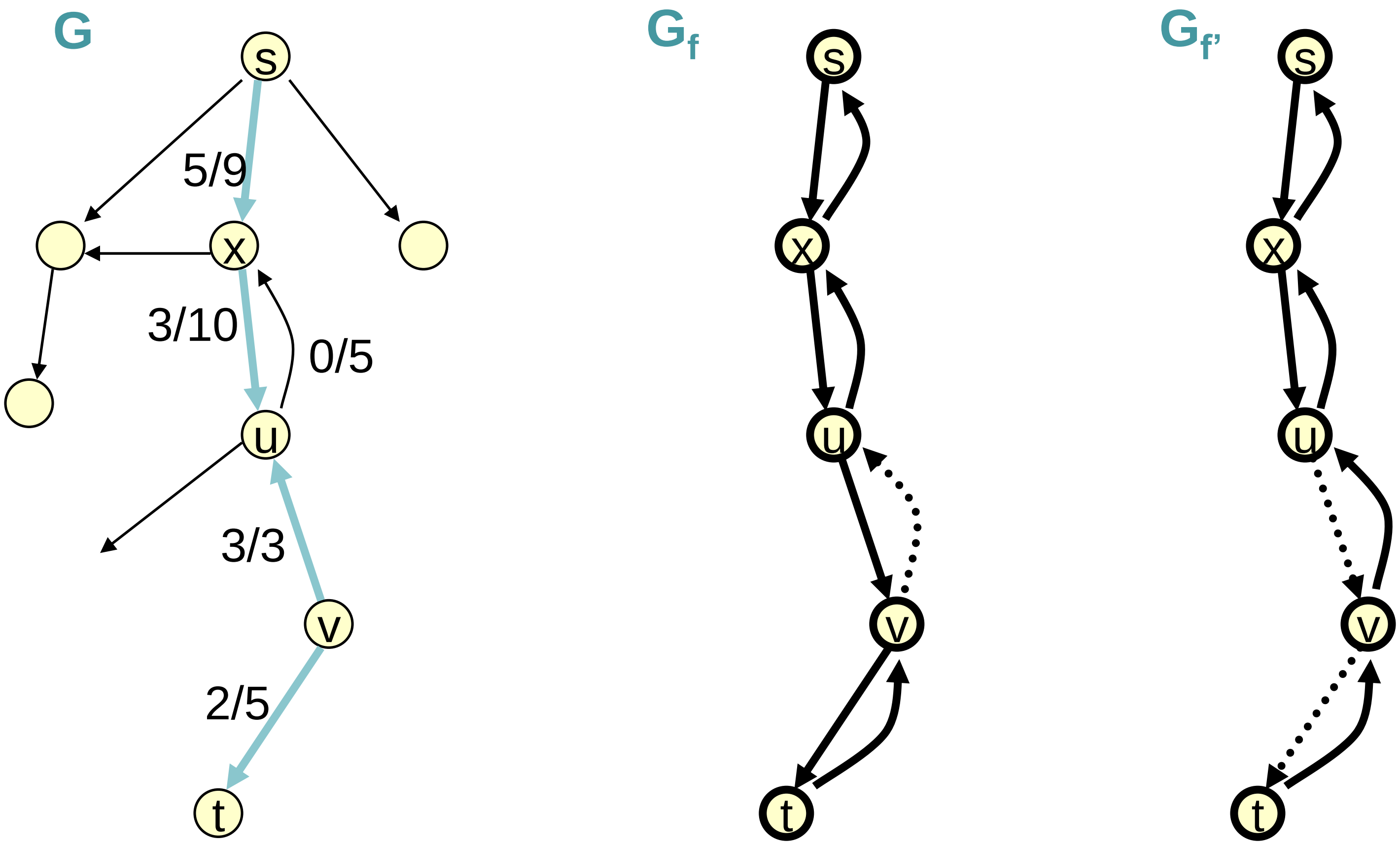
But, maxflow = 7.



Uri Zwick's non-terminating flow example, and three augmenting paths.

Edmonds-Karp Algorithm

- Use a shortest augmenting path
  (via Breadth First Search in residual graph)
- Time: $O(m^2 n)$.



Distance to s is non-decreasing.

Let f be a flow, G_f the residual graph, and P a shortest augmenting path. Then no vertex is closer to s after augmentation along P.

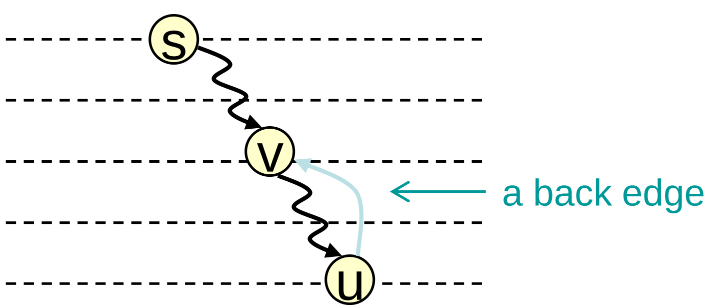
Proof: Augmentation along P only

- deletes forward edges

  no new (hence no shorter) path created

- adds back edges that go to previous vertices along P

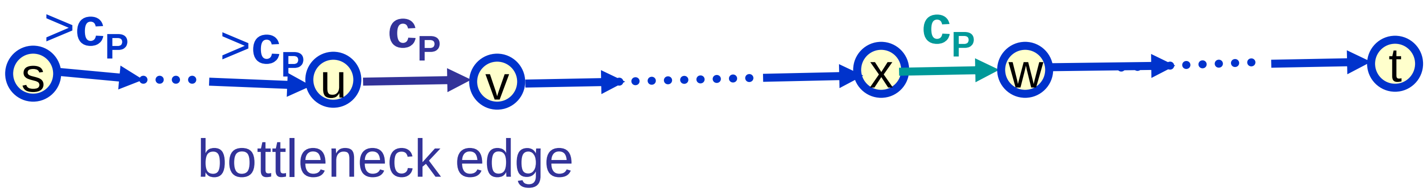
  BFS is unchanged, since v visited before (u, v) examined



Distance for bottleneck edges

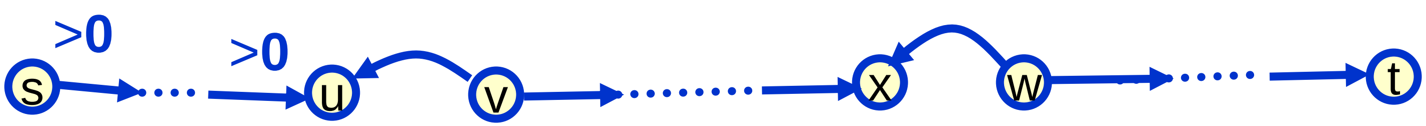
Let $d_f(s,v)$ be the distance from s to v on G_f.

Shortest s-t path **P** in $\mathbf{G_f}$

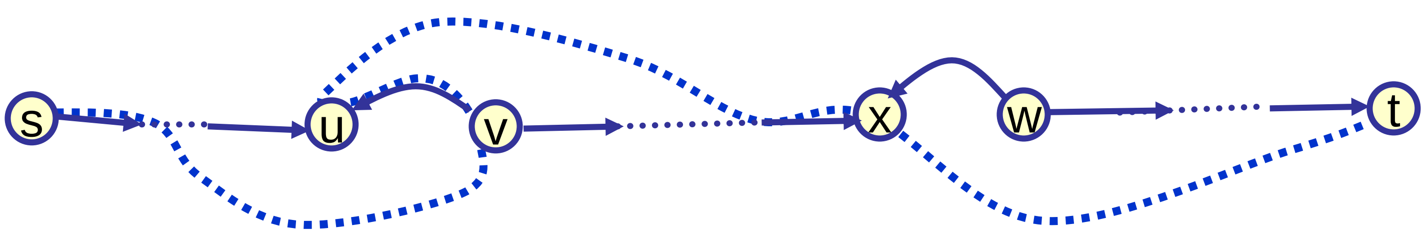


After augmenting along P

$d_f(s,v) = d_f(s,u) + 1$ since this is a shortest path



For (u,v) to be bottleneck again for some flow f'



$$d_{f'}(s,u) = d_{f'}(s,v) + 1 \geq d_f(s,v) + 1 = d_f(s,u) + 2$$

Theorem

Edmonds-Karp performs $O(mn)$ flow augmentations

Proof:

- Each step, some edge disappear from G_f.
  (Note however that some edge may reappear.)
- Any edge (u, v) disappears from G_f at most $n/2$ times.
  (because the distance increased by 2 every disappearance.)
- There are at most $mn/2$ disappearances.

Total time is $O(m^2n)$.

Maximum flow

Current Best (U = maximum capacity):

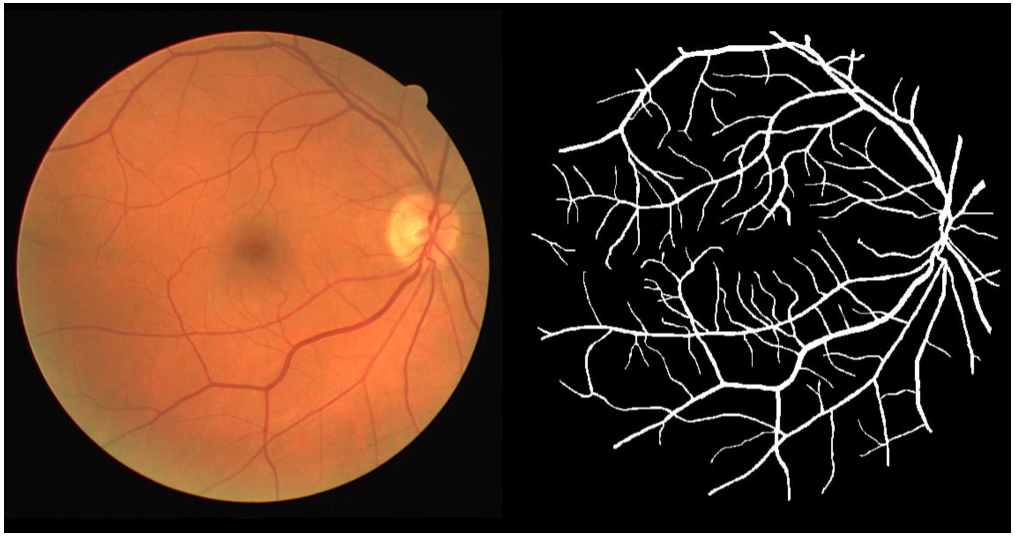
- $O((m+nF)\log^{O(1)}(nU))$ [Karger-Levine 02]
- $O(mn)$ [Orlin 13]
- $O(m\sqrt{n}\log^{O(1)}(nU))$ [Lee-Sidford 13]
- $O(m^{\frac{10}{7}}U^{\frac{1}{7}}\log^{O(1)}(nU))$ [Madry 13]
- $O((m+m^{\frac{3}{4}}n^{\frac{1}{4}}\sqrt{F})\log^{O(1)}(nU))$ [Sidford-Tian 17]

Image Segmentation

Image Segmentation

Given an image we want to separate foreground from background

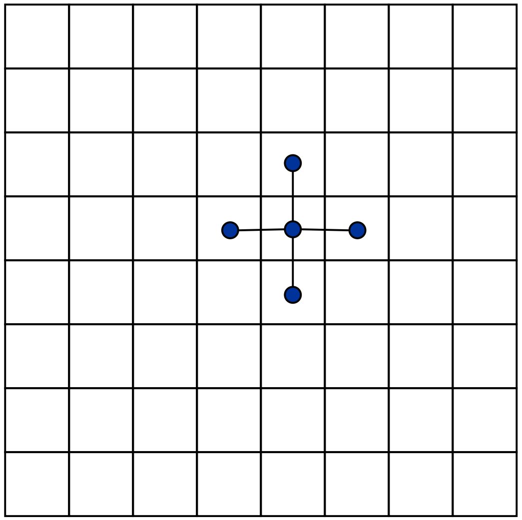
- Important problem in image processing.
- Divide image into coherent regions.



Foreground / background segmentation

Label each pixel as foreground/background.

- V = set of pixels, E = pairs of neighboring pixels.
- a_i is the original image.
- $a_i \gg 0$ means we prefer to label i in foreground.
- $p_{i,j} \geq 0$ is separation penalty for labeling one of i and j as foreground, and the other as background.



Goals:

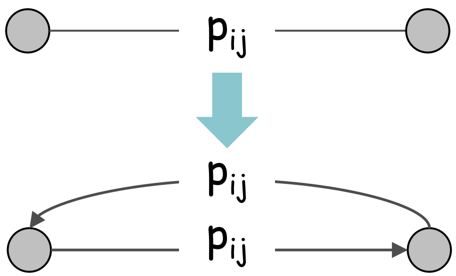
Find partition $(S, \overline{S})$ that minimizes:

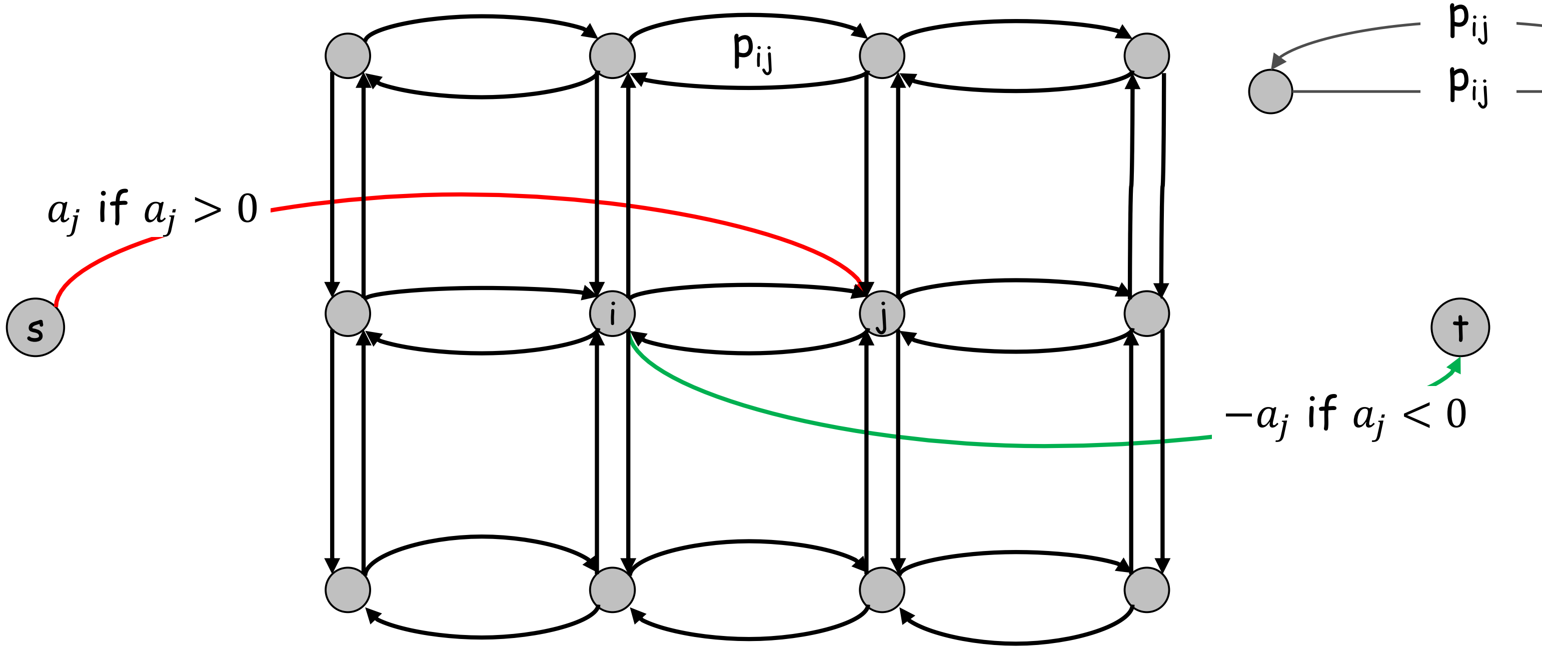
$$-\sum_{i \in S} a_i + \sum_{\substack{(i,j) \in E \\ i \in S, j \in \overline{S}}} p_{i,j}$$

where S is the foreground.

Min cut Formulation

$G' = (V', E')$.
Add s to correspond to foreground;
Add t to correspond to background;
Use two anti-parallel edges instead of undirected edge.





G'

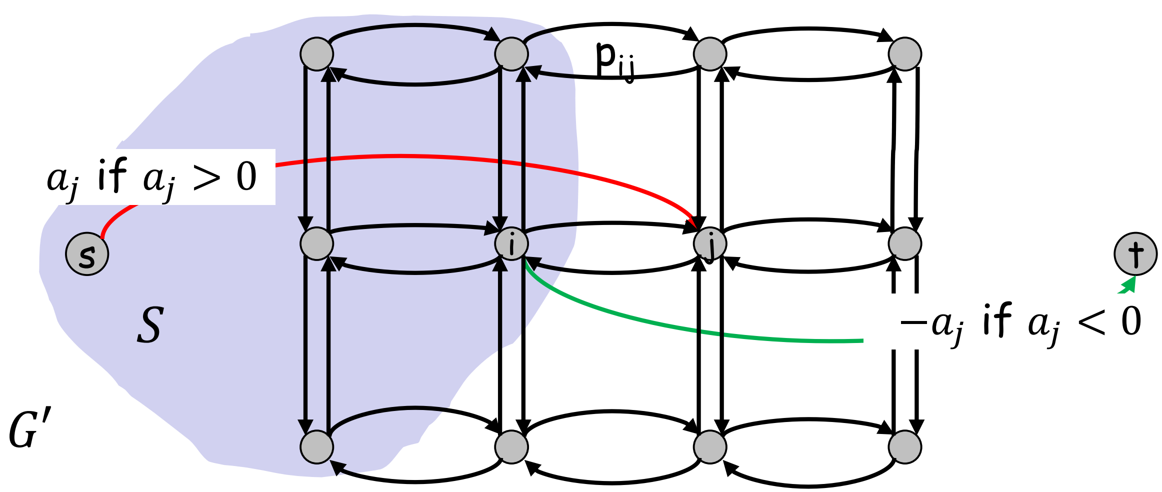
Min cut Formulation (cont'd)

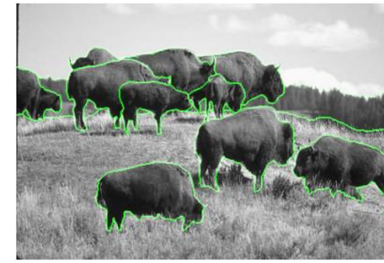
- Consider min cut $(S, \overline{S})$ in G'. (S = foreground.)

$$cap(S, \overline{S}) = \sum_{i \in S} -a_i \, 1_{a_i < 0} + \sum_{i \in \overline{S}} a_i \, 1_{a_i > 0} + \sum_{\substack{(i,j) \in E \\ i \in S, j \in \overline{S}}} p_{i,j}$$

$$= -\sum_{i \in S} a_i + \sum_{i \in S} a_i \, 1_{a_i > 0} + \sum_{i \in \overline{S}} a_i \, 1_{a_i > 0} + \sum_{\dots} p_{i,j}$$
$$= -\sum_{i \in S} a_i + \sum_i a_i + \sum_{\dots} p_{i,j}$$
$$= -\sum_{i \in S} a_i + \sum_{\dots} p_{i,j} + \text{constant}$$

Precisely, what we want to minimize.

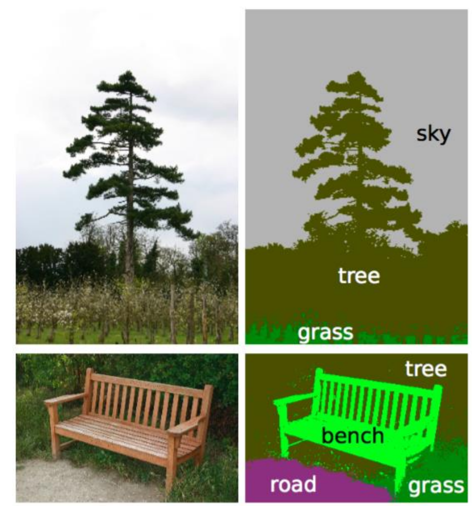




Reality

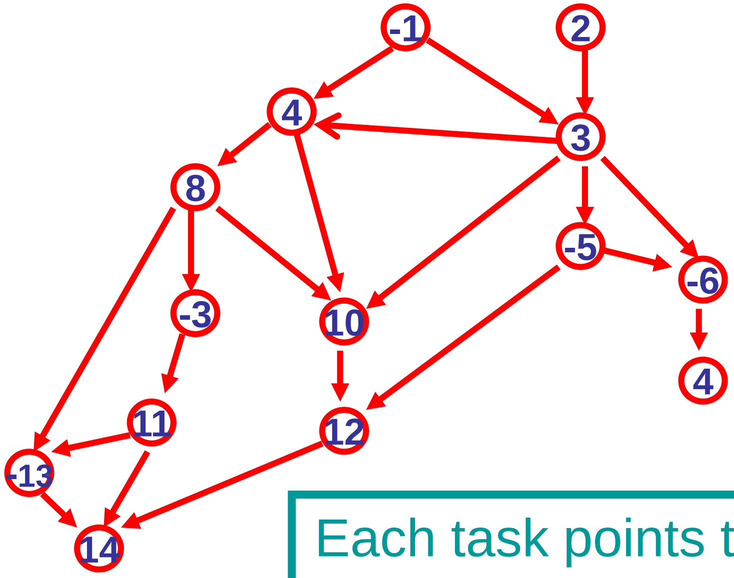
- The main difficulty is to come up with a good model.
- May want to have human interaction.

- Segmentation may be real-valued instead of {0,1}.
- There are many more than 1 objects.
- May need labeling.
- Augmenting path is not great for GPU.



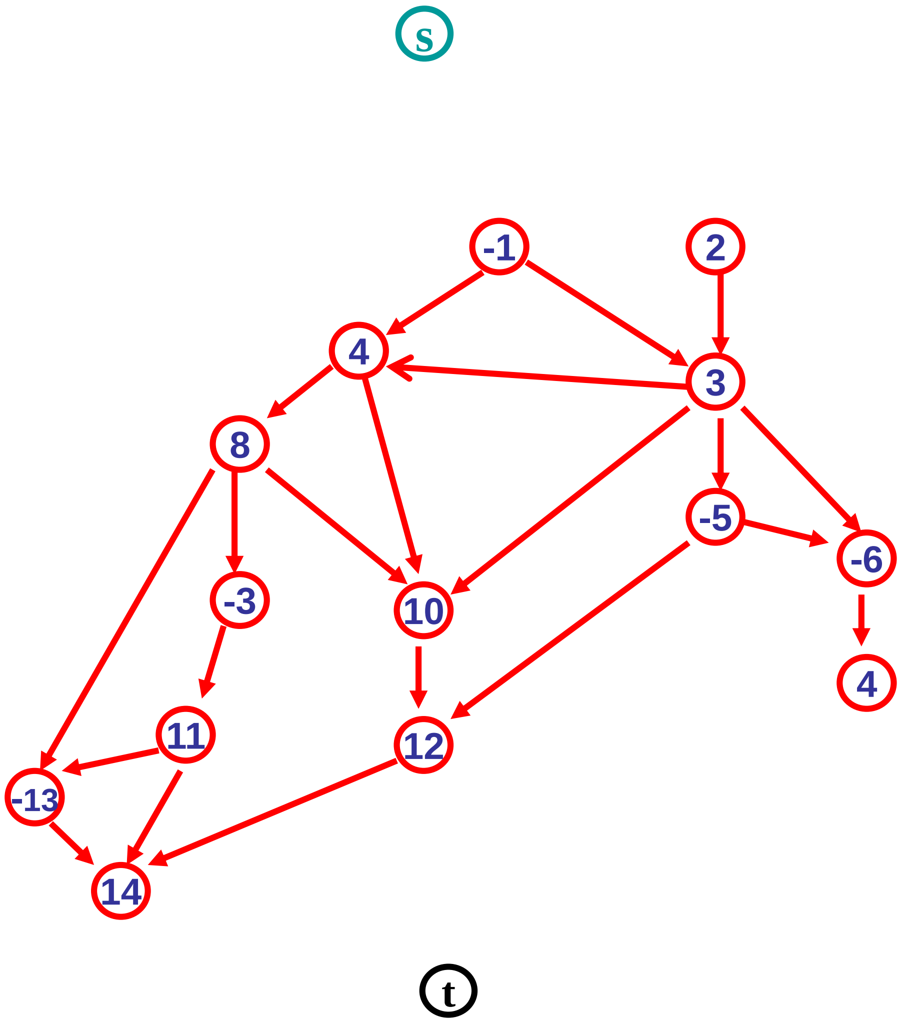
Project Selection

- Given a DAG $G=(V,E)$ representing
  precedence constraints on tasks
  (a task points to its **predecessors**)
  a profit value $p(v)$ for each task $v\in V$
  (may be positive or negative)
- Find a set $A\subseteq V$ of tasks that is closed under predecessors,
  (i.e. if $(u,v)\in E$ and $u\in A$ then $v\in A$)
  that maximizes $\text{Profit}(A)=\Sigma_{v\in A}\, p(v)$



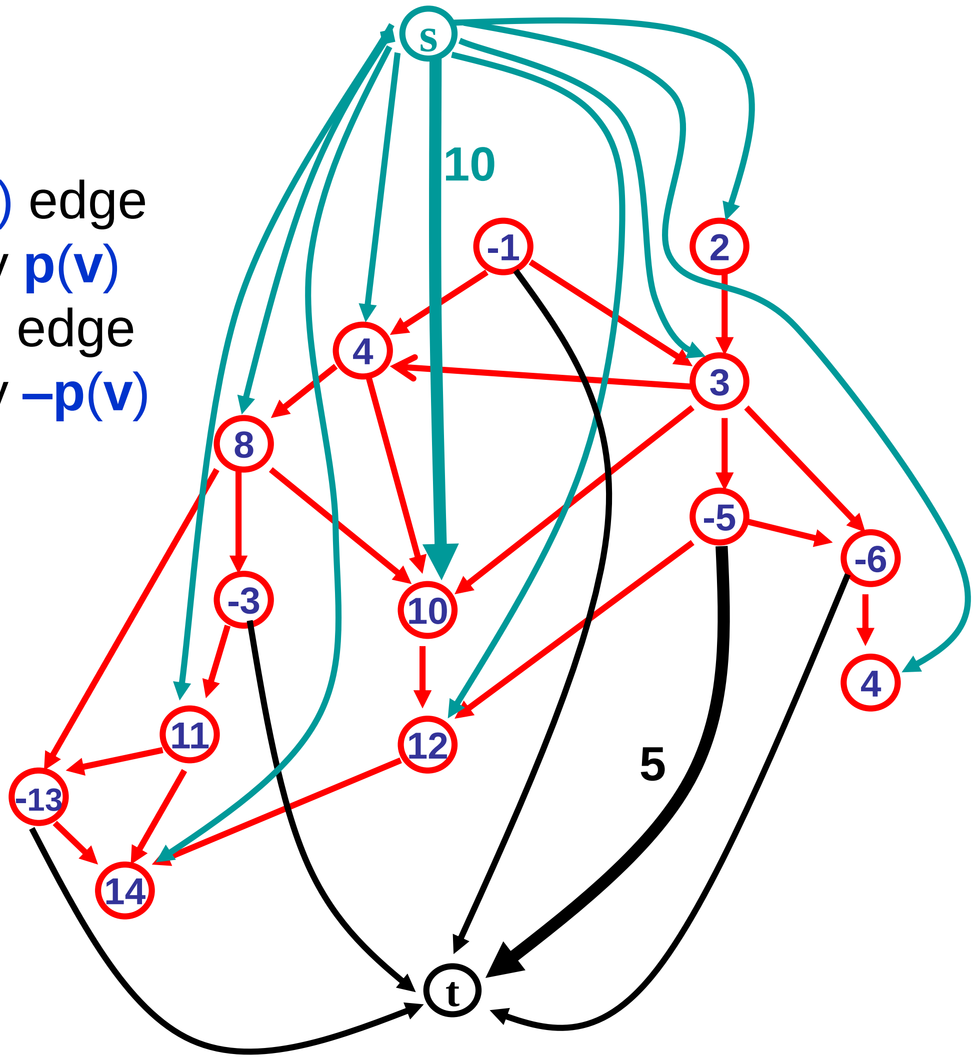
Each task points to its predecessor tasks

Extended Graph



Extended Graph G'

For each vertex **v**
If **p(v)≥0** add **(s,v)** edge
with capacity **p(v)**
If **p(v)<0** add **(v,t)** edge
with capacity **–p(v)**



Extended Graph G'

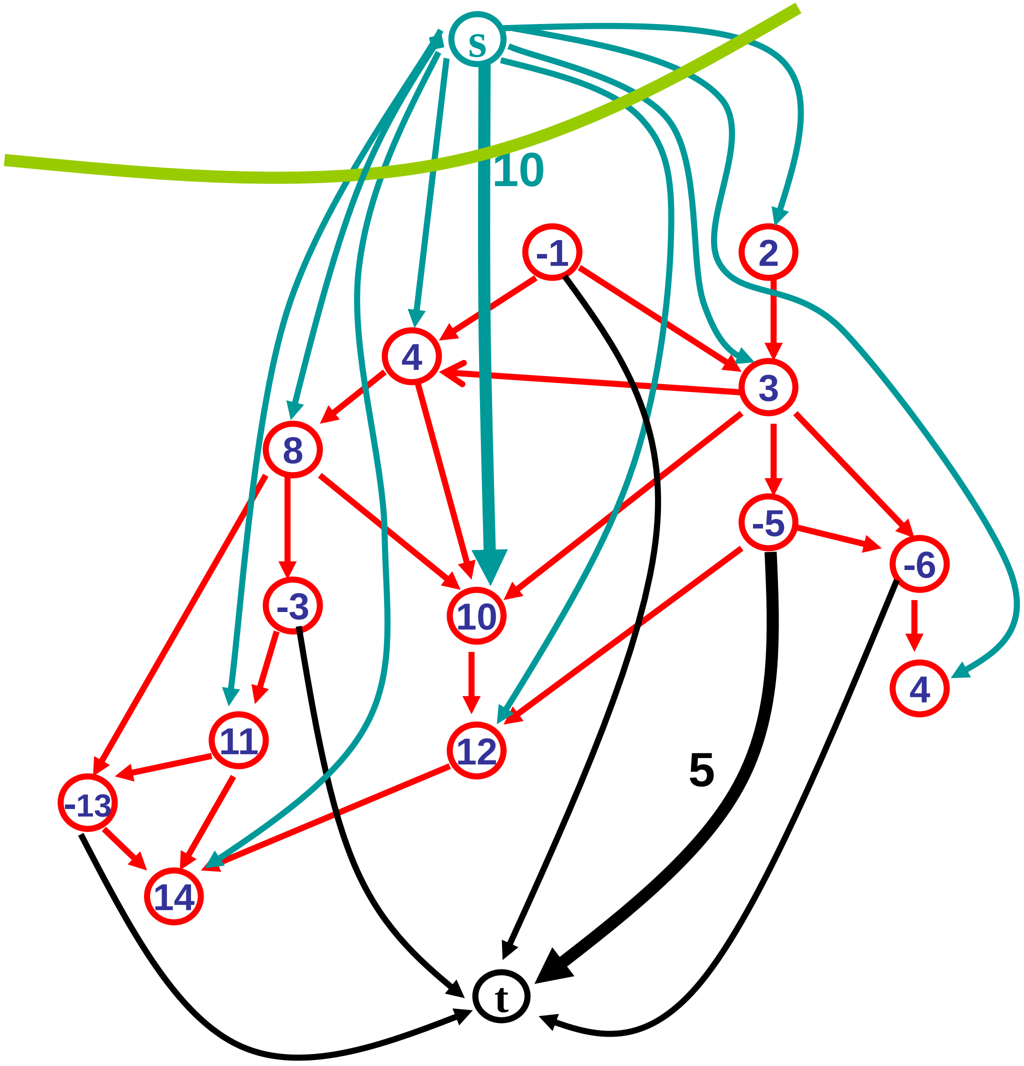
- Want: Set capacities on edges of **G** so that for minimum **s**-**t**-cut $(S,\bar{S})$ in **G'**, the set **A**=**S**-{**s**}
  - satisfies precedence constraints
  - has maximum possible profit in **G**
- Cut capacity with **S**={**s**} is just **C**=$\Sigma_{\mathbf{v}:\ \mathbf{p(v)}\geq 0}$ **p**(**v**)

  Profit(**A**) ≤ **C** for any set **A**
- To satisfy constraints, don't want any original edges of **G** going forward across the minimum cut

  That would correspond to a task in **A**=**S**-{**s**} that had a predecessor not in **A**=**S**-{**s**}
- Set capacity of each of the edges of **G** to $+\infty$.

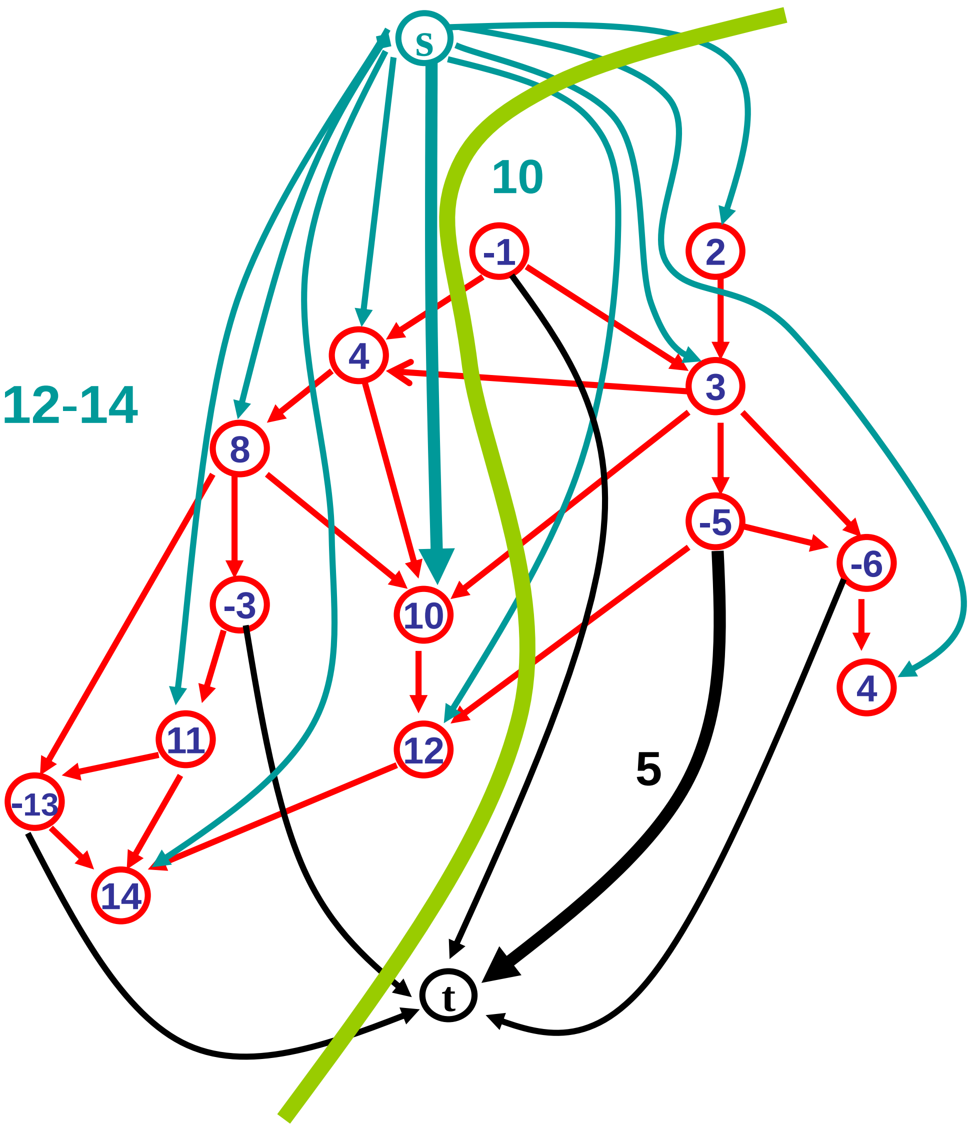
Extended Graph G'

Capacity $C = \Sigma_{v:\ p(v) \geq 0}\ p(v)$



Extended Graph G'

Cut value
=13+3+2+3+4
=13+3
  +C-4-8-10-11-12-14



Project Selection

- **Claim** Any **s**-**t**-cut $(S, \bar{S})$ in **G'** such that **A**=**S**-{**s**} satisfies
  - precedence constraints and
  - has capacity c(**S**,**T**)=**C** - $\Sigma_{v \in A}$ **p**(**v**) = **C** - Profit(**A**)

- **Corollary** A minimum cut $(S, \bar{S})$ in **G'** yields an optimal solution **A**=**S**-{**s**} to the project selection problem

- **Algorithm** Compute maximum flow **f** in **G'**, find the set **S** of nodes reachable from **s** in **G'**$_f$ and return **S**-{**s**}

Proof of Claim

- $A=S-\{s\}$ satisfies precedence constraints

  No edge of G crosses forward out of A since those edges have capacity $+\infty$

  Only forward edges cut are of the form

  $$(v,t) \text{ for } v\in A \text{ or } (s,v) \text{ for } v\notin A$$

  The (v,t) edges for $v\in A$ contribute

  $$\sum_{v\in A:p(v)<0} -p(v) = -\sum_{v\in A:p(v)<0} p(v)$$

  The (s,v) edges for $v\notin A$ contribute

  $$\sum_{v\notin A:\ p(v)\geq 0} p(v)=C-\sum_{v\in A:\ p(v)\geq 0} p(v)$$

  Therefore the total capacity of the cut is

  $$c(S,T) = C - \sum_{v\in A} p(v) = C-\text{Profit}(A)$$