CSE 421

Theory Application of Maxflow

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Maximum Matching Problem

Given an undirected graph $G = (V, E)$. A set $M \subseteq E$ is a matching if each node appears in at most 1 edge in $M$. Goal: find a matching with largest cardinality.
Bipartite Matching Problem

Given an undirected bipartite graph \( G = (X \cup Y, E) \)
A set \( M \subseteq E \) is a matching if each node appears in at most 1 edge in \( M \).
Goal: find a matching with largest cardinality.

\[
\begin{array}{c}
1 & 1' \\
2 & 2' \\
3 & 3' \\
4 & 4' \\
5 & 5'
\end{array}
\]

\( X \) \hspace{1cm} \( Y \)
Create digraph $H$ as follows:

- Orient all edges from $X$ to $Y$, and assign infinite (or unit) capacity.
- Add source $s$, and **unit** capacity edges from $s$ to each node in $L$.
- Add sink $t$, and **unit** capacity edges from each node in $R$ to $t$. 
Bipartite Matching: Proof of Correctness

**Thm.** Max cardinality matching in $G = \text{value of max flow in } H$.

**Proof.** Matching value $\leq \text{maxflow value}$

Given max matching $M$ of cardinality $k$.

Consider flow $f$ that sends 1 unit along each of $k$ edges of $M$.

$f$ is a flow, and has cardinality $k$. □
Bipartite Matching: Proof of Correctness

**Thm.** Max cardinality matching in $G = \text{value of max flow in } H$.

**Proof.** (matching val $\geq$ maxflow val) Let $f$ be a max flow in $H$ of value $k$. Integrality theorem $\Rightarrow k$ is integral and we can assume $f$ is 0-1.

Consider $M = \text{set of edges from } X \text{ to } Y \text{ with } f(e) = 1$.

- each node in $X$ and $Y$ participates in at most one edge in $M$.
- $|M| = k$: consider s-t cut $(s \cup X, t \cup Y)$
Perfect Bipartite Matching
Perfect Bipartite Matching

Def. A matching $M \subseteq E$ is perfect if each node appears in exactly one edge in $M$.

Q. When does a bipartite graph have a perfect matching?

Structure of bipartite graphs with perfect matchings:
• Clearly we must have $|X| = |Y|$.
• What other conditions are necessary?
• What conditions are sufficient?
Perfect Bipartite Matching: $N(S)$

**Def.** Let $S$ be a subset of nodes, and let $N(S)$ be the set of nodes adjacent to nodes in $S$.

**Observation.** If a bipartite graph $G$ has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq X$.

**Proof.** Each $v \in S$ has to be matched to a unique node in $N(S)$.
Marriage Theorem

Thm: [Frobenius 1917, Hall 1935]
Let $G = (X \cup Y, E)$ be a bipartite graph with $|X| = |Y|$. Then, $G$ has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq X$.

Proof. $\Rightarrow$
This was the previous observation.
If $|N(S)| < |S|$ for some $S$, then there is no perfect matching.
Marriage Theorem

**Pf.** \( \exists S \subseteq X \) s.t., \(|N(S)| < |S| \iff G \) does not a perfect matching
Formulate as a max-flow and let \((A, B)\) be the min \(s - t\) cut
\(G\) has no perfect matching \(\Rightarrow v(f^*) < |X|\). So, \(cap(A, B) < |X|\)
Define \(X_A = X \cap A, X_B = X \cap B, Y_A = Y \cap A\)
Then, \(cap(A, B) = |X_B| + |Y_A|\)
Since min-cut does not use \(\infty\) edges, \(N(X_A) \subseteq Y_A\)
\(|N(X_A)| \leq |Y_A| = cap(A, B) - |X_B| = cap(A, B) - |X| + |X_A| < |X_A|\)
Bipartite Matching Running Time

Which max flow algorithm to use for bipartite matching?

Generic augmenting path: $O(m \text{ val}(f^*)) = O(mn)$.

Shortest augmenting path: $O(m\sqrt{n})$. [Even and Tarjan 75, Karzanov 73]

Current record: $O(m^{10/7} \log^{O(1)} n)$ [Madry 13]

😢 I tried few years on improving it 😢
Edge Disjoint Paths
Edge Disjoint Paths Problem

Given a digraph $G = (V, E)$ and two nodes $s$ and $t$, find the max number of edge-disjoint $s - t$ paths.

**Def.** Two paths are edge-disjoint if they have no edge in common.

**Ex:** communication networks.
Max Flow Formulation

Assign a unit capacitary to every edge. Find Max flow from $s$ to $t$.

**Thm.** Max number edge-disjoint s-t paths equals max flow value.

**Proof.** # of disjoint path $\leq$ maxflow value

Suppose there are $k$ edge-disjoint paths $P_1, \ldots, P_k$.

Set $f(e) = 1$ if $e$ participates in some path $P_i$; else set $f(e) = 0$.

Since paths are edge-disjoint, $f$ is a flow of value $k$. ▪
Thm. Max number edge-disjoint s-t paths equals max flow value.
Pf. # of disjoint path $\geq$ maxflow val
Suppose max flow value is $k$
Integrality theorem $\Rightarrow$ there exists 0-1 flow $f$ of value $k$.
Consider edge $(s, u)$ with $f(s, u) = 1$.
• by conservation, there exists an edge $(u, v)$ with $f(u, v) = 1$
• continue until reach $t$, always choosing a new edge
This produces $k$ (not necessarily simple) edge-disjoint paths.

We can return to $u$ so we can have cycles. But we can eliminate cycles if desired
Network Connectivity
Network Connectivity

Given a digraph $G = (V, E)$ and two nodes $s$ and $t$, find the minimum number of edges whose removal disconnects $t$ from $s$.

**Def.** A set of edges $F \subseteq E$ disconnects $t$ from $s$ if all $s - t$ paths use at least one edge in $F$.

**Ex:** In testing network reliability
Thm. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Proof. ≤
Suppose the removal of \( F \subseteq E \) disconnects \( t \) from \( s \), and \( |F| = k \). All s-t paths use at least one edge of \( F \). Hence, the number of edge-disjoint paths is at most \( k \).
Network Connectivity using Min Cut

Thm. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Proof. ≥
Suppose there are $k$ edge disjoint paths from $s$ to $t$
So, Max flow is $k$
So, there is a $s-t$ cut $(A,B)$ s.t., $cap(A,B) = k$
Let $F$ be the edges out of $A$. So, $|F| = k$.
If we remove $F$ we disconnect $t$ from $s$. 