CSE 421

Theory Application of Maxflow

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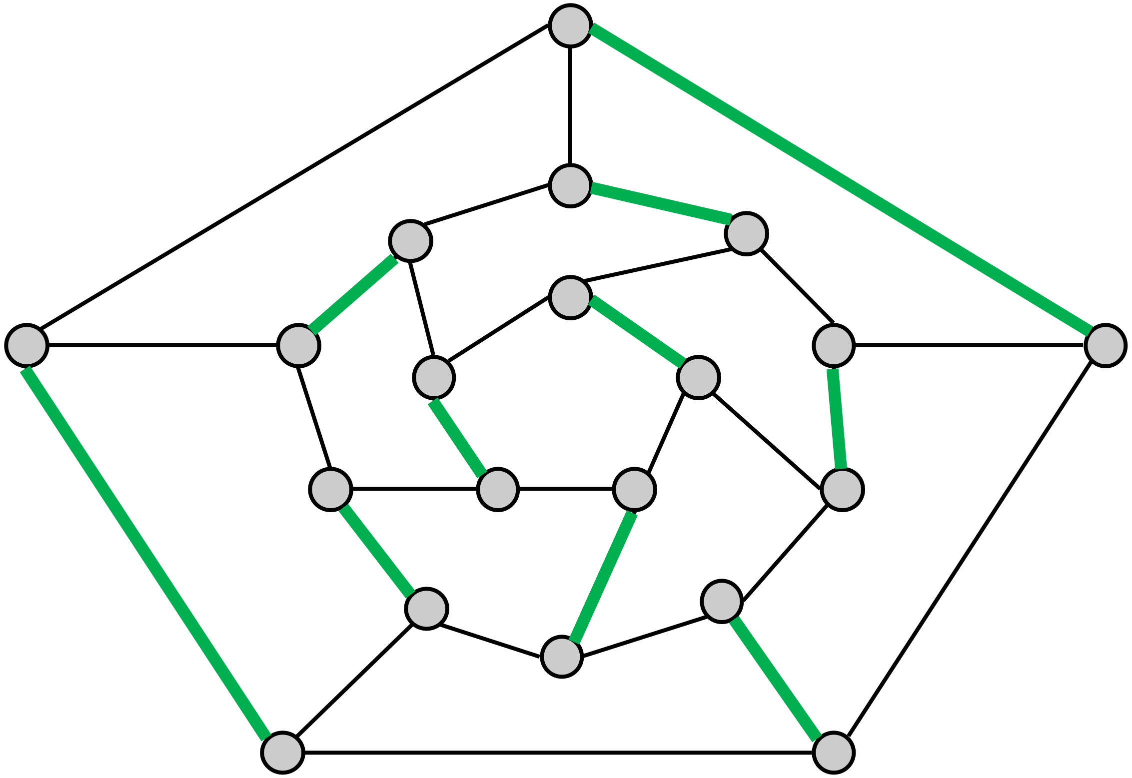
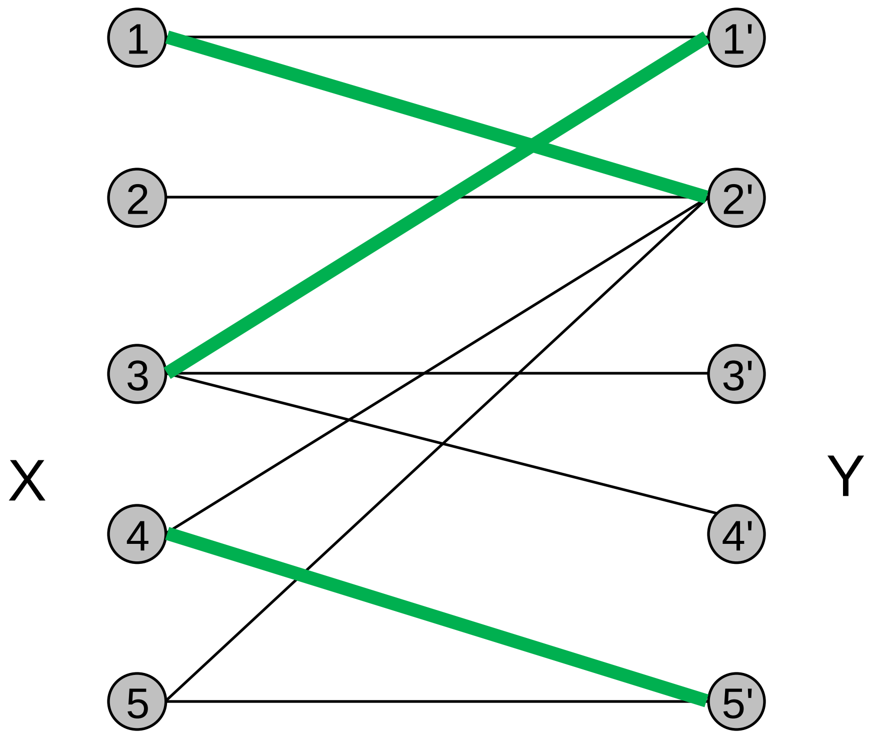
Maximum Matching Problem

Given an undirected graph $G = (V, E)$.
A set $M \subseteq E$ is a matching if each node appears in at most 1 edge in M.
Goal: find a matching with largest cardinality.



Bipartite Matching Problem

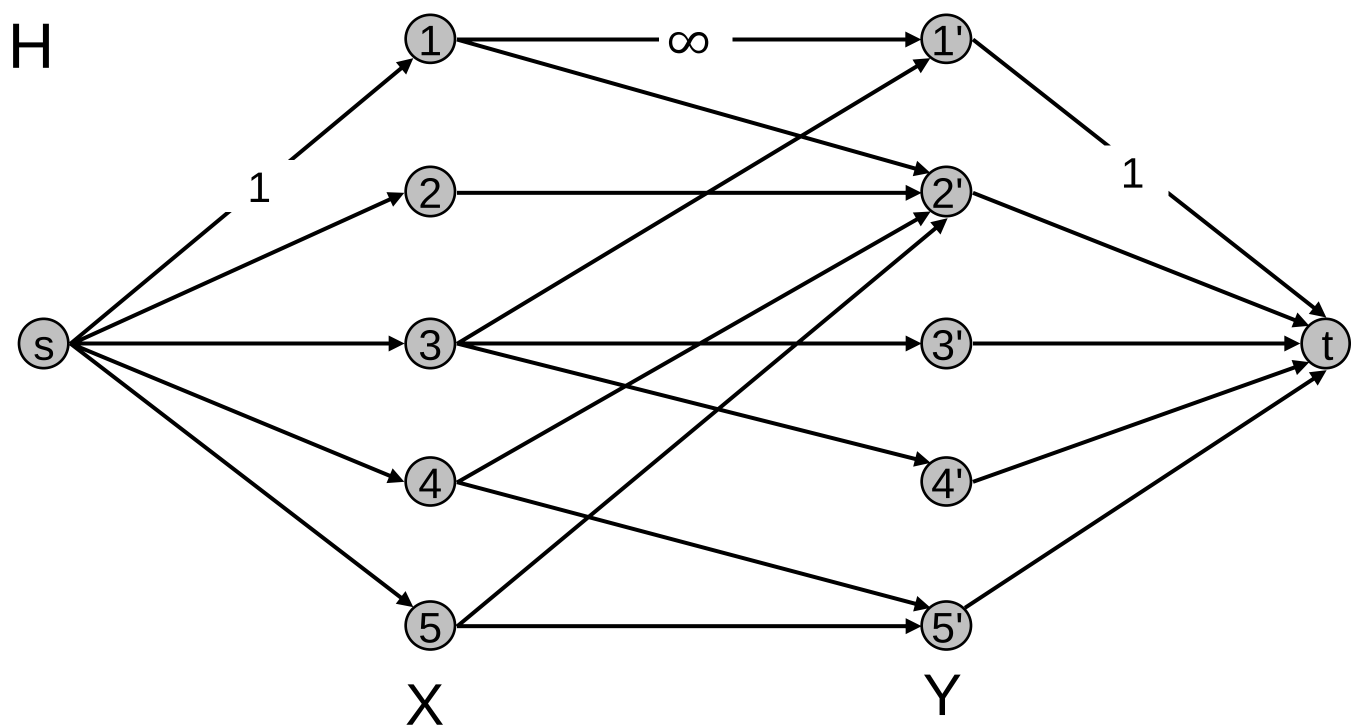
Given an undirected bipartite graph $G = (X \cup Y, E)$
A set $M \subseteq E$ is a matching if each node appears in at most 1 edge in M.
Goal: find a matching with largest cardinality.



Bipartite Matching using Max Flow

Create digraph H as follows:

- Orient all edges from X to Y, and assign infinite (or unit) capacity.
- Add source s, and unit capacity edges from s to each node in L.
- Add sink t, and unit capacity edges from each node in R to t.



Bipartite Matching: Proof of Correctness

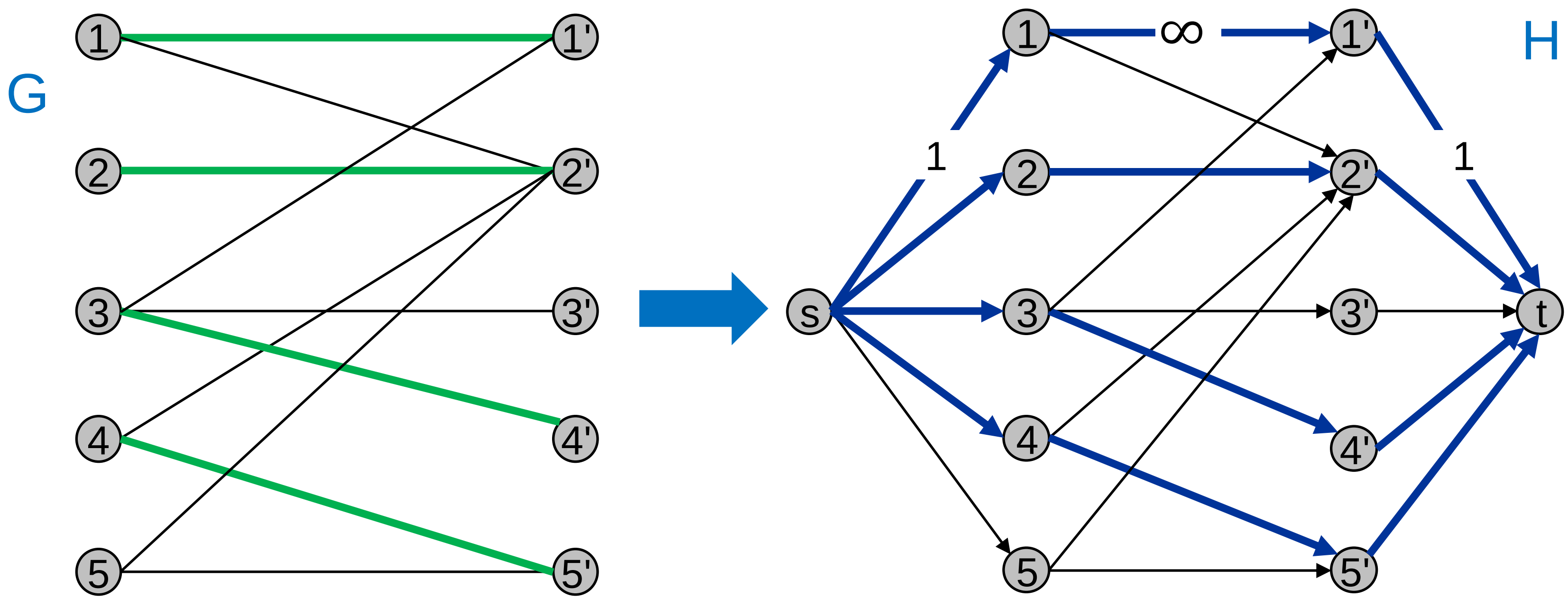
Thm. Max cardinality matching in G = value of max flow in H.

Proof. Matching value ≤ maxflow value

Given max matching M of cardinality k.

Consider flow f that sends 1 unit along each of k edges of M.

f is a flow, and has cardinality k. ▪



Bipartite Matching: Proof of Correctness

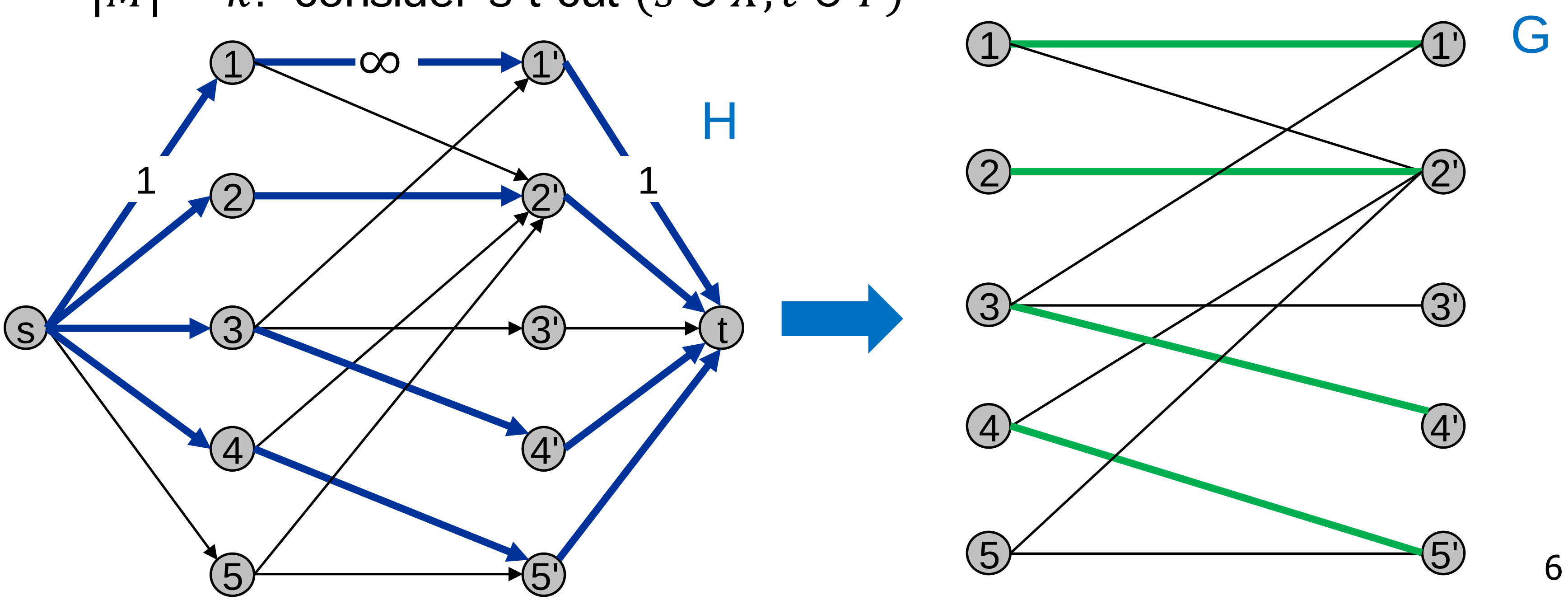
Thm. Max cardinality matching in G = value of max flow in H.

Proof. (matching val ≥ maxflow val) Let f be a max flow in H of value k.

Integrality theorem $\Rightarrow$ k is integral and we can assume f is 0-1.

Consider M = set of edges from X to Y with $f(e) = 1$.

- each node in X and Y participates in at most one edge in M.
- $|M| = k$: consider s-t cut $(s \cup X, t \cup Y)$



Perfect Bipartite Matching

Perfect Bipartite Matching

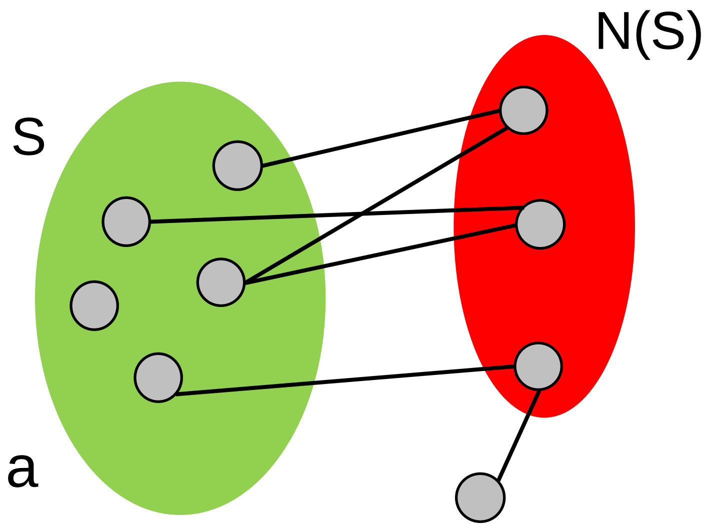
Def. A matching $M \subseteq E$ is perfect if each node appears in exactly one edge in M.

Q. When does a bipartite graph have a perfect matching?

Structure of bipartite graphs with perfect matchings:

- Clearly we must have $|X| = |Y|$.
- What other conditions are necessary?
- What conditions are sufficient?

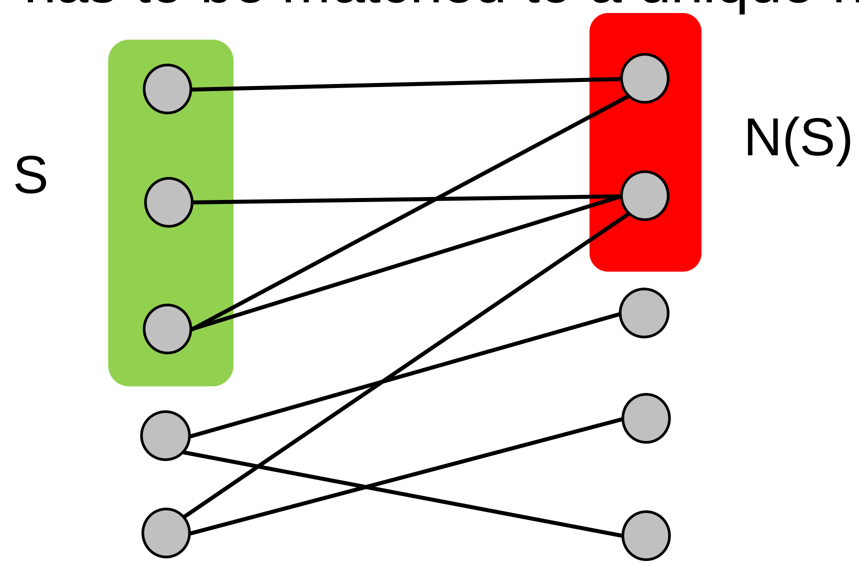
Perfect Bipartite Matching: N(S)



Def. Let S be a subset of nodes, and let $N(S)$ be the set of nodes adjacent to nodes in S.

Observation. If a bipartite graph G has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq X$.

Proof. Each $v \in S$ has to be matched to a unique node in $N(S)$.



Marriage Theorem

Thm: [Frobenius 1917, Hall 1935]

Let $G = (X \cup Y, E)$ be a bipartite graph with $|X| = |Y|$.

Then, G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq X$.

Proof. $\Rightarrow$

This was the previous observation.

If $|N(S)| < |S|$ for some S, then there is no perfect matching.

Marriage Theorem

Pf. $\exists S \subseteq X$ s.t., $|N(S)| < |S| \Leftarrow G$ does not a perfect matching

Formulate as a max-flow and let (A, B) be the min $s - t$ cut

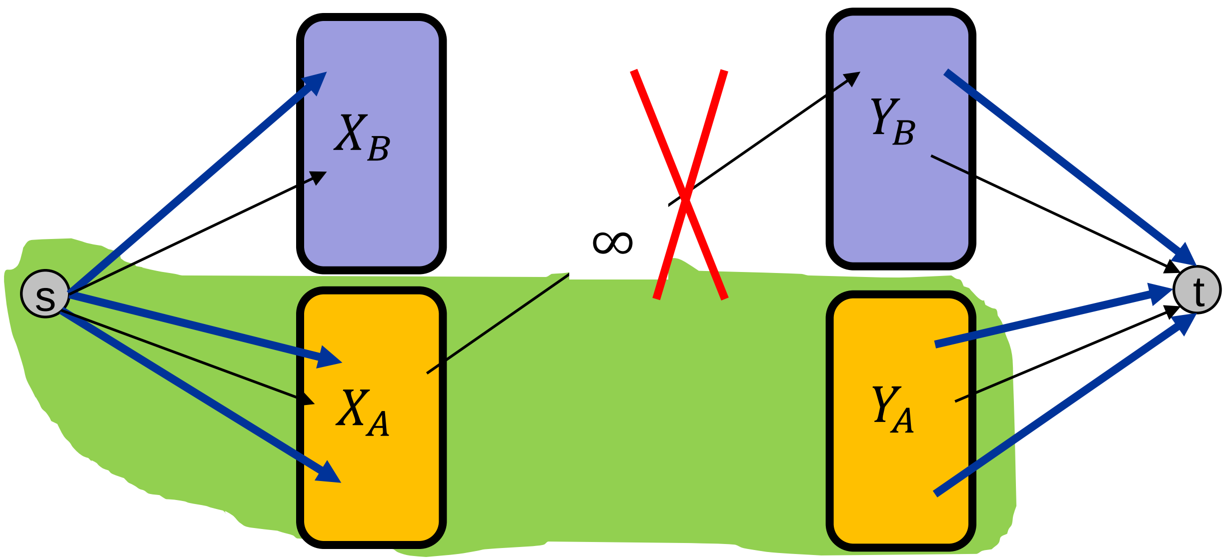
G has no perfect matching $\Rightarrow v(f^*) < |X|$. So, $cap(A, B) < |X|$

Define $X_A = X \cap A, X_B = X \cap B, Y_A = Y \cap A$

Then, $cap(A, B) = |X_B| + |Y_A|$

Since min-cut does not use ∞ edges, $N(X_A) \subseteq Y_A$

$|N(X_A)| \leq |Y_A| = cap(A, B) - |X_B| = cap(A, B) - |X| + |X_A| < |X_A|$



Bipartite Matching Running Time

Which max flow algorithm to use for bipartite matching?

- Generic augmenting path: $O(m\ val(f^*)) = O(mn)$.
- Shortest augmenting path: $O(m\sqrt{n})$. [Even and Tarjan 75, Karzanov 73]
- Current record: $O(m^{10/7} \log^{O(1)} n)$ [Madry 13]
- ☹ I tried few years on improving it ☹

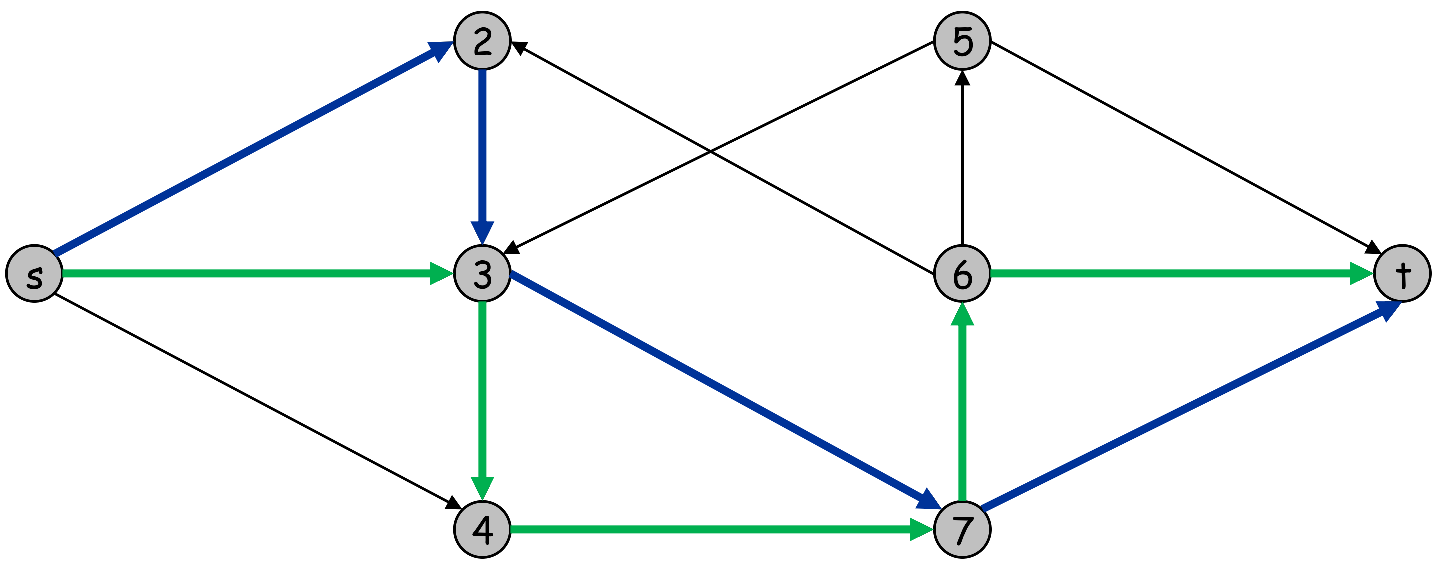
Edge Disjoint Paths

Edge Disjoint Paths Problem

Given a digraph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint $s - t$ paths.

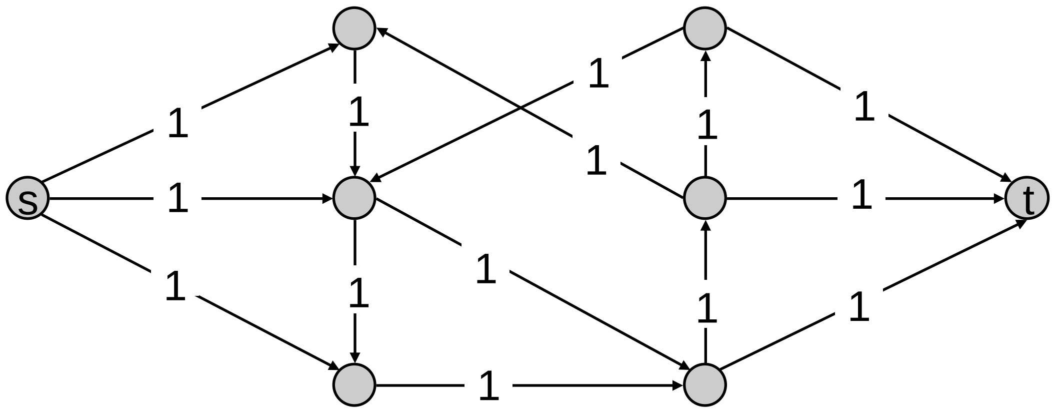
Def. Two paths are edge-disjoint if they have no edge in common.

Ex: communication networks.



Max Flow Formulation

Assign a unit capacitary to every edge. Find Max flow from s to t.



Thm. Max number edge-disjoint s-t paths equals max flow value.

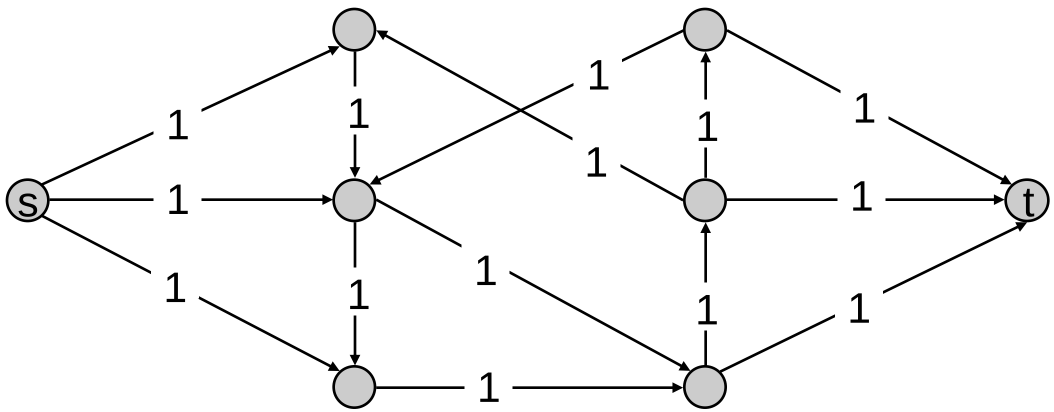
Proof. # of disjoint path ≤ maxflow value

Suppose there are k edge-disjoint paths $P_1, \dots, P_k$.

Set $f(e) = 1$ if e participates in some path P_i ; else set $f(e) = 0$.

Since paths are edge-disjoint, f is a flow of value k. ▪

Max Flow Formulation



Thm. Max number edge-disjoint s-t paths equals max flow value.

Pf. # of disjoint path ≥ maxflow val Suppose max flow value is k

Integrality theorem $\Rightarrow$ there exists 0-1 flow f of value k.

Consider edge (s, u) with $f(s, u) = 1$.

- by conservation, there exists an edge (u, v) with $f(u, v) = 1$
- continue until reach t, always choosing a new edge

This produces k (not necessarily simple) edge-disjoint paths. ▪

We can return to u so we can have cycles. But we can eliminate cycles if desired

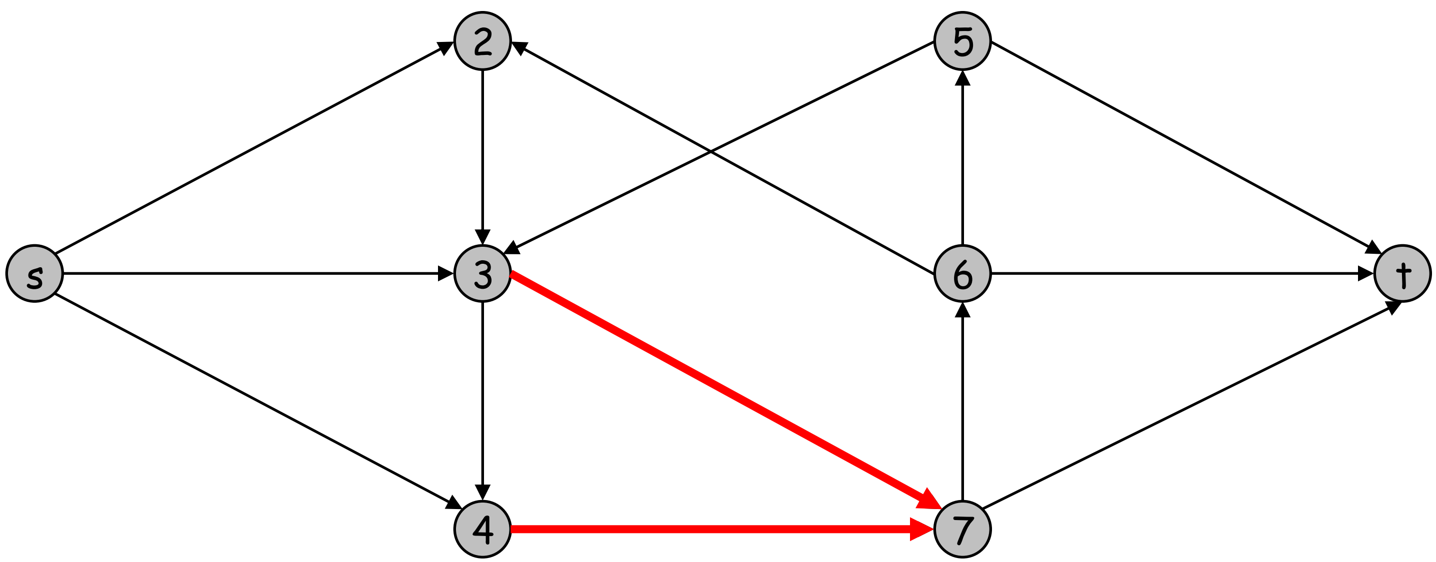
Network Connectivity

Network Connectivity

Given a digraph $G = (V, E)$ and two nodes s and t, find min number of edges whose removal disconnects t from s.

Def. A set of edges $F \subseteq E$ disconnects t from s if all $s - t$ paths uses at least one edge in F.

Ex: In testing network reliability



Network Connectivity using Min Cut

Thm. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Proof. $\leq$

Suppose the removal of $F \subseteq E$ disconnects t from s, and $|F| = k$.

All s-t paths use at least one edge of F. Hence, the number of edge-disjoint paths is at most k.

Network Connectivity using Min Cut

Thm. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Proof. $\geq$

Suppose there are k edge disjoint paths from s to t

So, Max flow is k

So, there is a $s-t$ cut (A,B) s.t.,

$$cap(A, B) = k$$

Let F be the edges out of A. So,

$$|F| = k.$$

If we remove F we disconnect t from s.