CSE 421

Divide and Conquer / Median

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Median

Selecting k-th smallest

Problem: Given numbers $x_1, \dots, x_n$ and an integer $1 \leq k \leq n$ output the k-th smallest number

$$\text{Sel}(\{x_1, \dots, x_n\}, k)$$

A simple algorithm: Sort the numbers in time $O(n \log n)$ then return the k-th smallest in the array.

Can we do better?

Yes, in time $O(n)$ if $k = 1$ or $k = 2$.

Can we do $O(n)$ for all possible values of k?

An Idea

Choose a number w from $x_1, \ldots, x_n$

Define

- $S_<(w) = \{x_i : x_i < w\}$
- $S_=(w) = \{x_i : x_i = w\}$
- $S_>(w) = \{x_i : x_i > w\}$

Can be computed in linear time

Solve the problem recursively as follows:

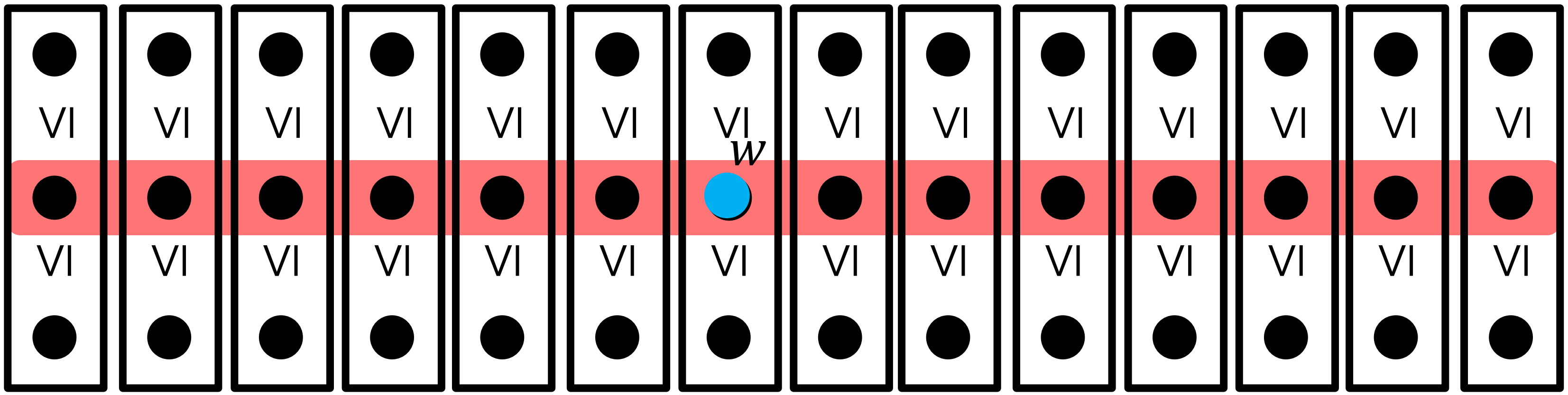
- If $k \le |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \le |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - |S_<(w)| - |S_=(w)|)$

Ideally want $|S_<(w)|, |S_>(w)| \le n/2$. In this case ALG runs in $O(n) + O\left(\frac{n}{2}\right) + O\left(\frac{n}{4}\right) + \cdots + O(1) = O(n)$.

How to choose w?

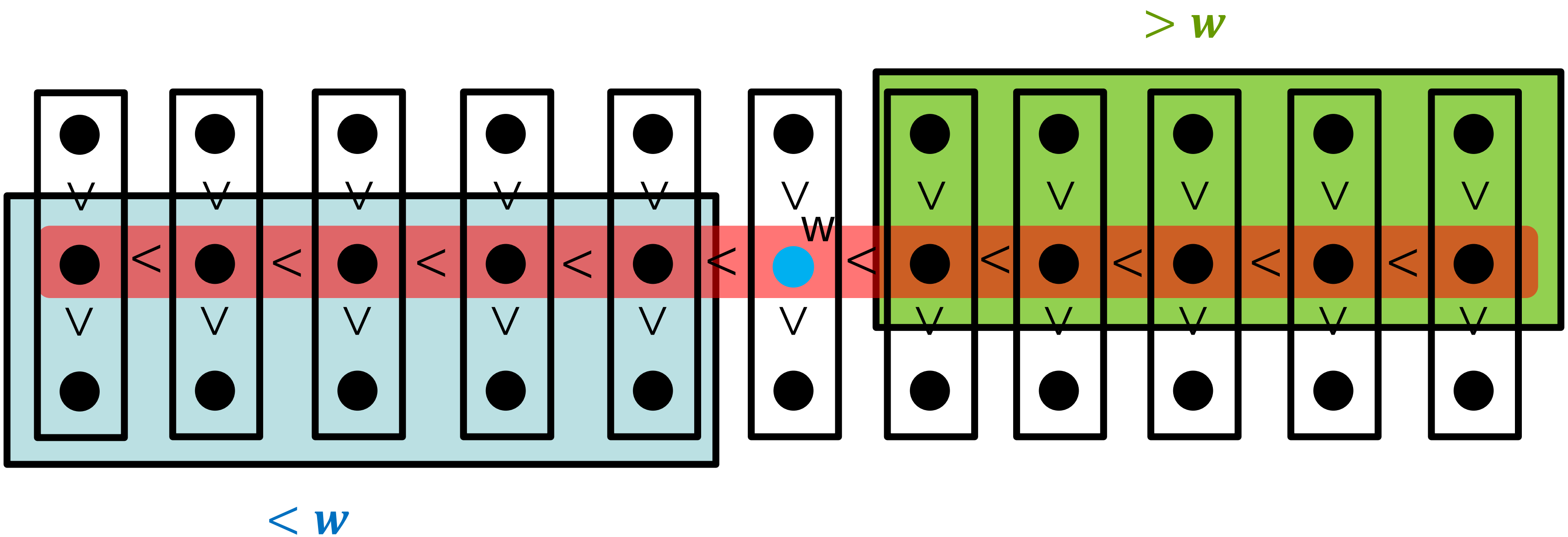
Suppose we choose w uniformly at random
similar to the pivot in quicksort.
Then, $\mathbb{E}[|S_<(w)|] = \mathbb{E}[|S_>(w)|] = n/2$. Algorithm runs in $O(n)$ in expectation.
Can we get $O(n)$ running time deterministically?

- Partition numbers into sets of size 3.
- Sort each set (takes O(n))
- $w = Sel(\textcolor{red}{midpoints}, n/6)$



Assume all numbers are distinct for simplicity.

How to lower bound $|S_<(w)|, |S_>(w)|$?



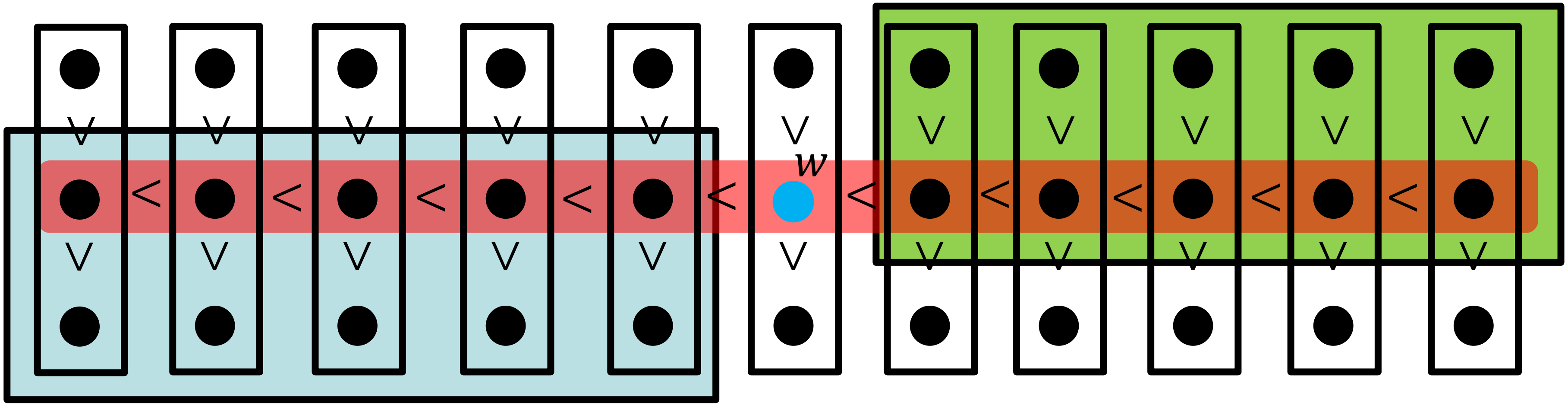
- $|S_<(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$
- $|S_>(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$.

$$\frac{n}{3} \leq |S_<(w)|, |S_>(w)| \leq \frac{2n}{3}$$

So, what is the running time?

Assume all numbers are distinct for simplicity.

Asymptotic Running Time?



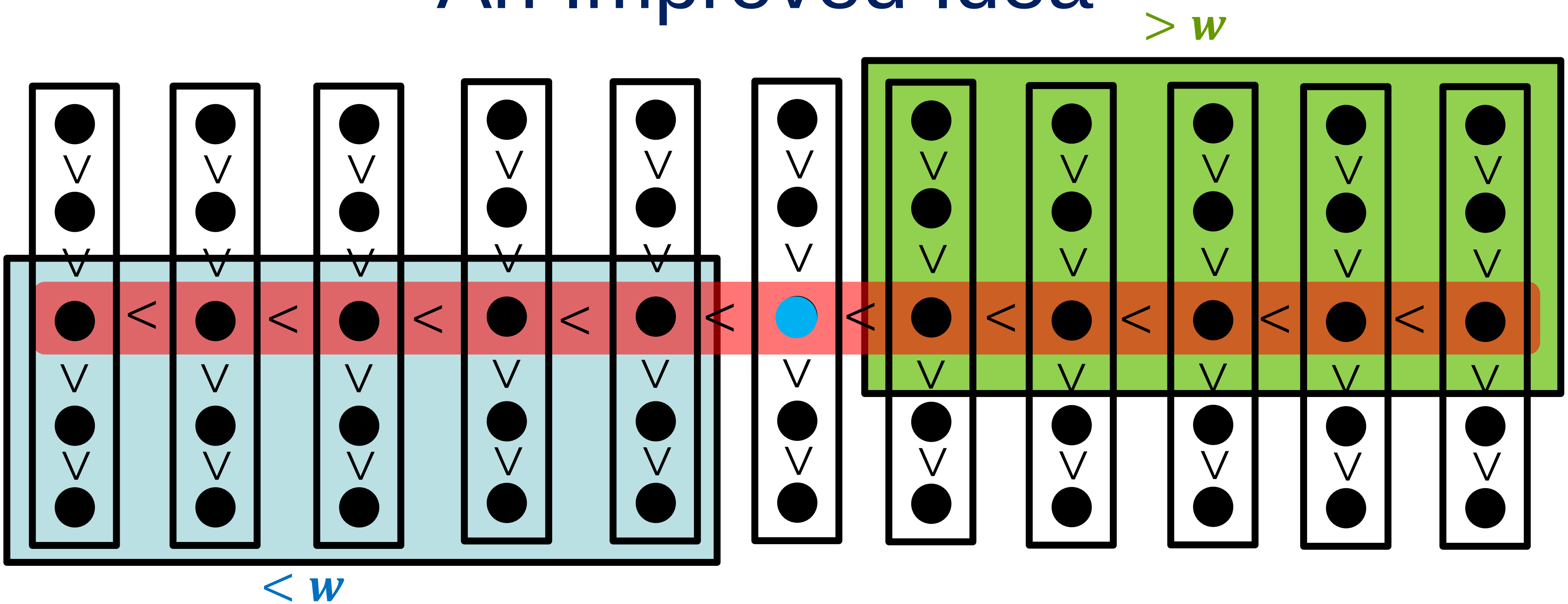
- If $k \le |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \le |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - |S_<(w)| - |S_=(w)|)$

$O(n \log n)$ again?
So, what is the point?

Where $\frac{n}{3} \le |S_<(w)|, |S_>(w)| \le \frac{2n}{3}$

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + O(n) \Rightarrow T(n) = O(n \log n)$$

An Improved Idea



Partition into n/5 sets. Sort each set and set $w = Sel(\color{red}{midpoints}, n/10)$

- $|S_<(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$
- $|S_>(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$

$$\Rightarrow \quad \frac{3n}{10} \leq |S_<(w)|, |S_>(w)| \leq \frac{7n}{10}$$

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n) \Rightarrow T(n) = O(n)$$

An Improved Idea

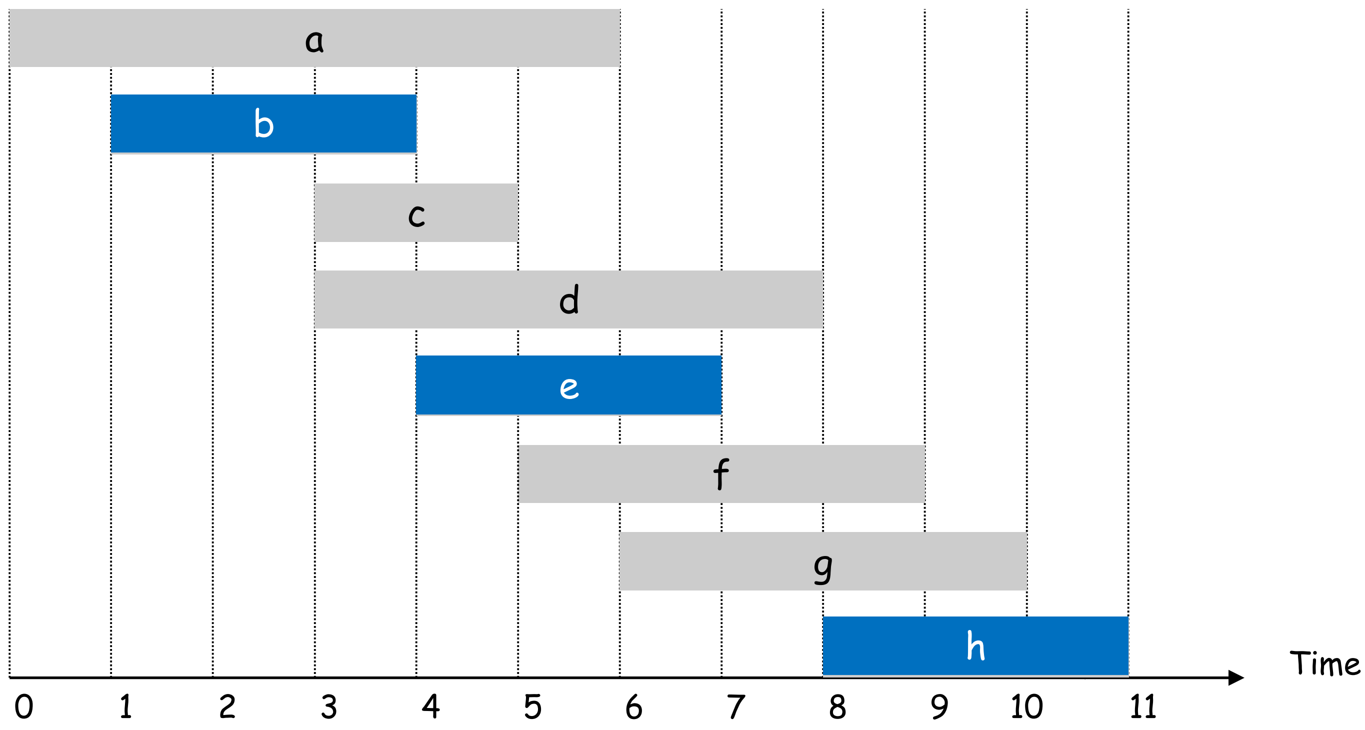
```
Sel(S, k) {
  $n \leftarrow |S|$
  If (n < ??) return ??
  Partition S into n/5 sets of size 5
  Sort each set of size 5 and let M be the set of medians, so |M|=n/5
  Let w=Sel(M,n/10)
  For i=1 to n{
    If $x_i < w$ add x to $S_<(w)$
    If $x_i > w$ add x to $S_>(w)$
    If $x_i = w$ add x to $S_=(w)$
  }
  If ($k \leq |S_<(w)|$)
    return Sel($S_<(w), k$)
  else if ($k \leq |S_<(w)| + |S_=(w)|$)
    return w;
  else
    return Sel($S_>(w), k - |S_<(w)| - |S_=(w)|$)
}
```

We can maintain each set in an array

Weighted Interval Scheduling

Interval Scheduling

- Job j starts at $s(j)$ and finishes at $f(j)$ and has weight w_j
- Two jobs compatible if they don't overlap.
- Goal: find maximum weight subset of mutually compatible jobs.

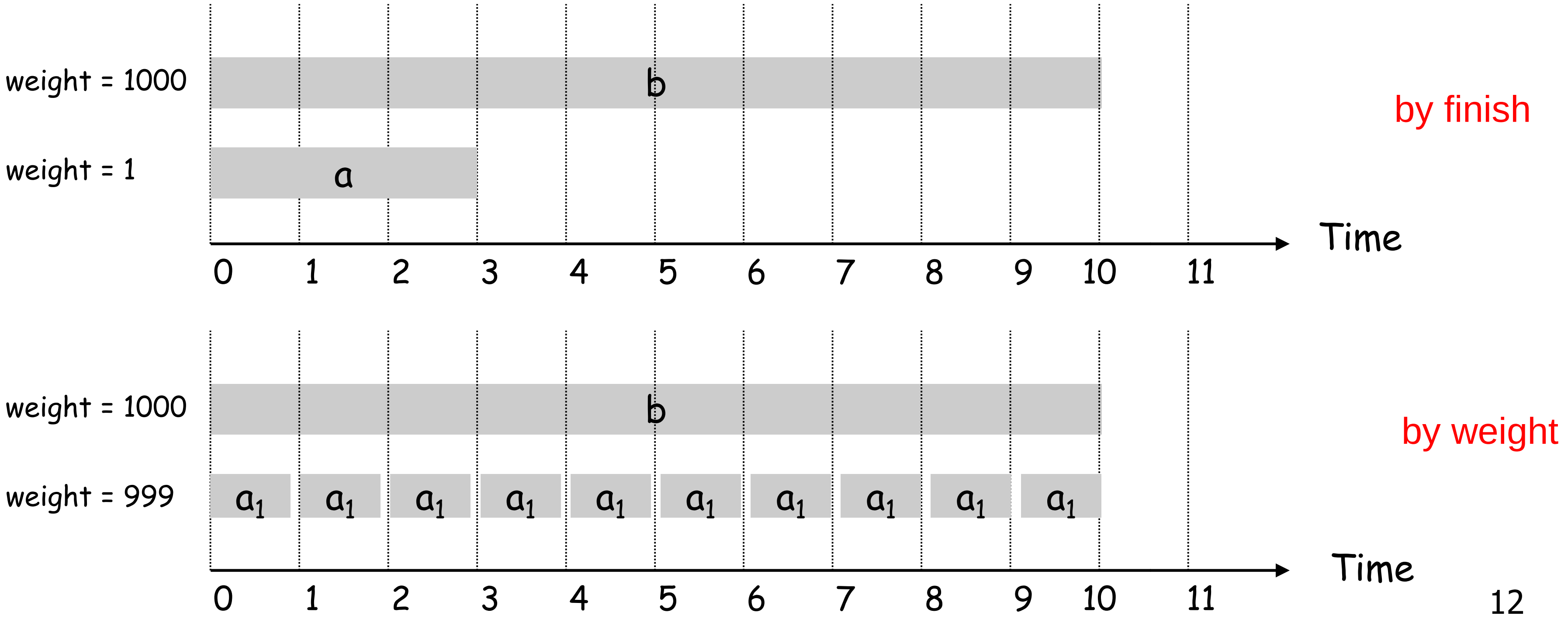


Unweighted Interval Scheduling: Review

Recall: Greedy algorithm works if all weights are 1:

- Consider jobs in ascending order of finishing time
- Add job to a subset if it is compatible with prev added jobs.

Observation: Greedy ALG fails spectacularly if arbitrary weights are allowed:



Weighted Job Scheduling by Induction

Suppose $1, \ldots, n$ are

This idea works for any Optimization problem.

For NP-hard problems there is no ordering to reduce # subproblems

IH: Suppose … bs.

IS: Goal: For

Case 1: Job n is no…

-- Then, just return OPT of $1, \ldots, n$ –

Take best of the two

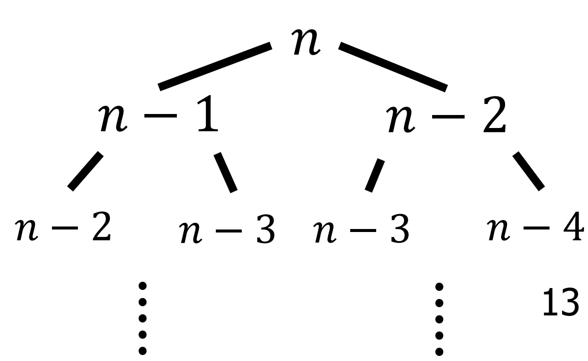
Case 2: Job n is in OPT.

-- Then, delete all jobs not compatible with n and recurse.

Q: Are we done?

A: No, How many subproblems are there?

Potentially 2^n all possible subsets of jobs.

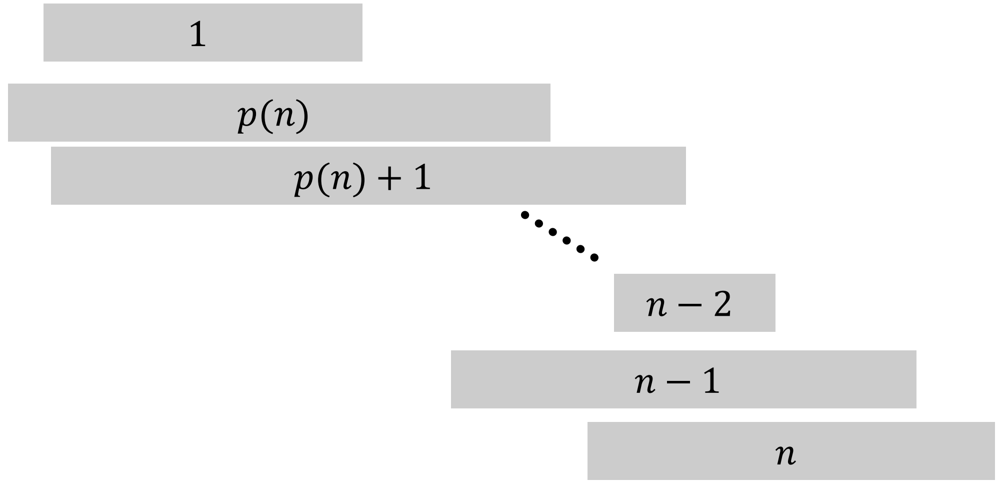


Sorting to Reduce Subproblems

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$
IS: For jobs $1, \ldots, n$ we want to compute OPT

Case 1: Suppose OPT has job n.

- So, all jobs i that are not compatible with n are not OPT
- Let $p(n)$ = largest index $i < n$ such that job i is compatible with n.
- Then, we just need to find OPT of $1, \ldots, p(n)$



Sorting to reduce Subproblems

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$
IS: For jobs $1, \dots, n$ we want to compute OPT

Case 1: Suppose OPT has job n.

- So, all jobs i that are not compatible with n are not OPT
- Let $p(n)$ = largest index $i < n$ such that job i is compatible with n.
- Then, we just need to find OPT of $1, \dots, p(n)$

Case 2: OPT does not select job n.

- Then, OPT is just the OPT of $1, \dots, n-1$

Take best of the two

Q: Have we made any progress (still reducing to two subproblems)?
A: Yes! This time every subproblem is of the form $1, \dots, i$ for some i
So, at most n possible subproblems.

Weighted Job Scheduling by Induction

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$

Def $OPT(j)$ denote the weight of OPT solution of $1, \ldots, j$

The most important part of a correct DP; It fixes IH

To solve $OPT(j)$:

Case 1: $OPT(j)$ has job j.

- So, all jobs i that are not compatible with j are not $OPT(j)$.
- Let $p(j) =$ largest index $i < j$ such that job i is compatible with j.
- So $OPT(j) = OPT(p(j)) + w_j$.

Case 2: $OPT(j)$ does not select job j.

- Then, $OPT(j) = OPT(j-1)$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

Algorithm

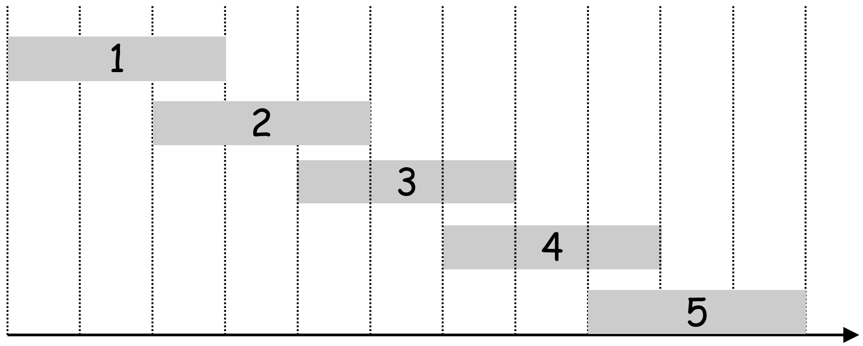
```
Input: n, s(1),…,s(n) and f(1),…,f(n) and w_1,…,w_n.

Sort jobs by finish times so that f(1) ≤ f(2) ≤ ⋯ f(n).

Compute p(1),p(2),…,p(n)

OPT(j) {
   if ( j = 0 )
      return 0
   else
      return max(w_j + OPT(p(j)), OPT(j − 1)).
}
```

Recursive Algorithm Fails

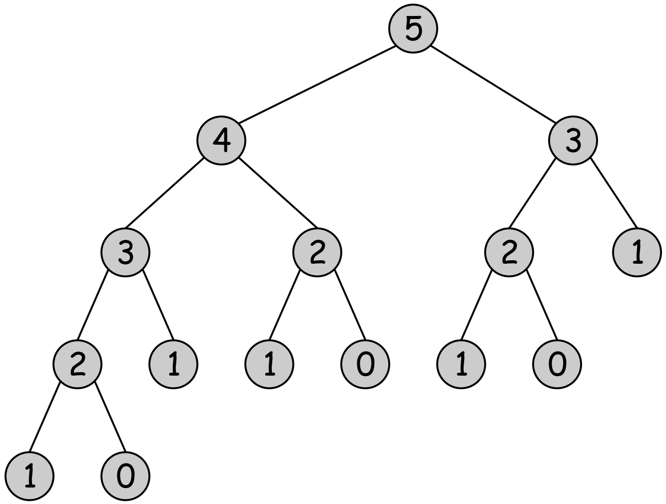
Even though we have only n subproblems, we do not store the solution to the subproblems

➢ So, we may re-solve the same problem many many times.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence



$p(1) = 0, p(j) = j - 2$



Algorithm with Memoization

Memorization. Compute and Store the solution of each sub-problem in a cache the first time that you face it. lookup as needed.

```
Input: n, s(1),...,s(n)  and f(1),...,f(n) and w_1,...,w_n.

Sort jobs by finish times so that f(1) ≤ f(2) ≤ ⋯ f(n).

Compute p(1),p(2),...,p(n)

for j = 1 to n
   M[j] = empty
M[0] = 0

OPT(j) {
   if (M[j] is empty)
      M[j] = max (w_j + OPT(p(j)), OPT(j − 1)).
   return M[j]
}
```

In practice, you may get stack overflow if $n \gg 10^6$ (depends on the language).

Bottom up Dynamic Programming

You can also avoid recursion

- recursion may be easier conceptually when you use induction

```
Input: n, s(1),...,s(n)  and f(1),...,f(n) and w_1,...,w_n.

Sort jobs by finish times so that f(1) ≤ f(2) ≤ ··· f(n).

Compute p(1),p(2),...,p(n)

OPT(j) {
   M[0] = 0
   for j = 1 to n
      M[j] = max(w_j + M[p(j)], M[j-1]).
}

Output M[n]
```

Claim: $M[j]$ is value of $OPT(j)$

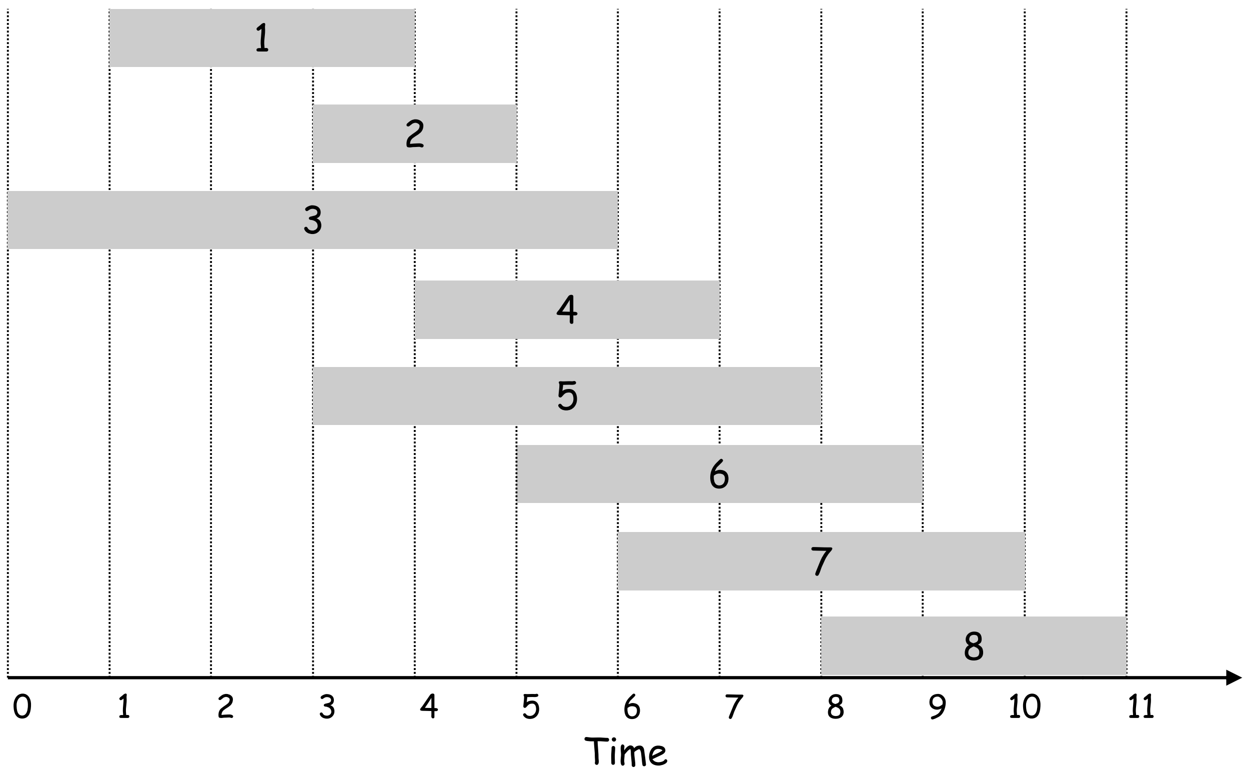
Timing: Easy. Main loop is $O(n)$; sorting is $O(n \log n)$.

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

$p(j)$ = largest index $i < j$ such that job i is compatible with j.



j	w_j	p(j)	OPT(j)
0			0
1	3	0	
2	4	0	
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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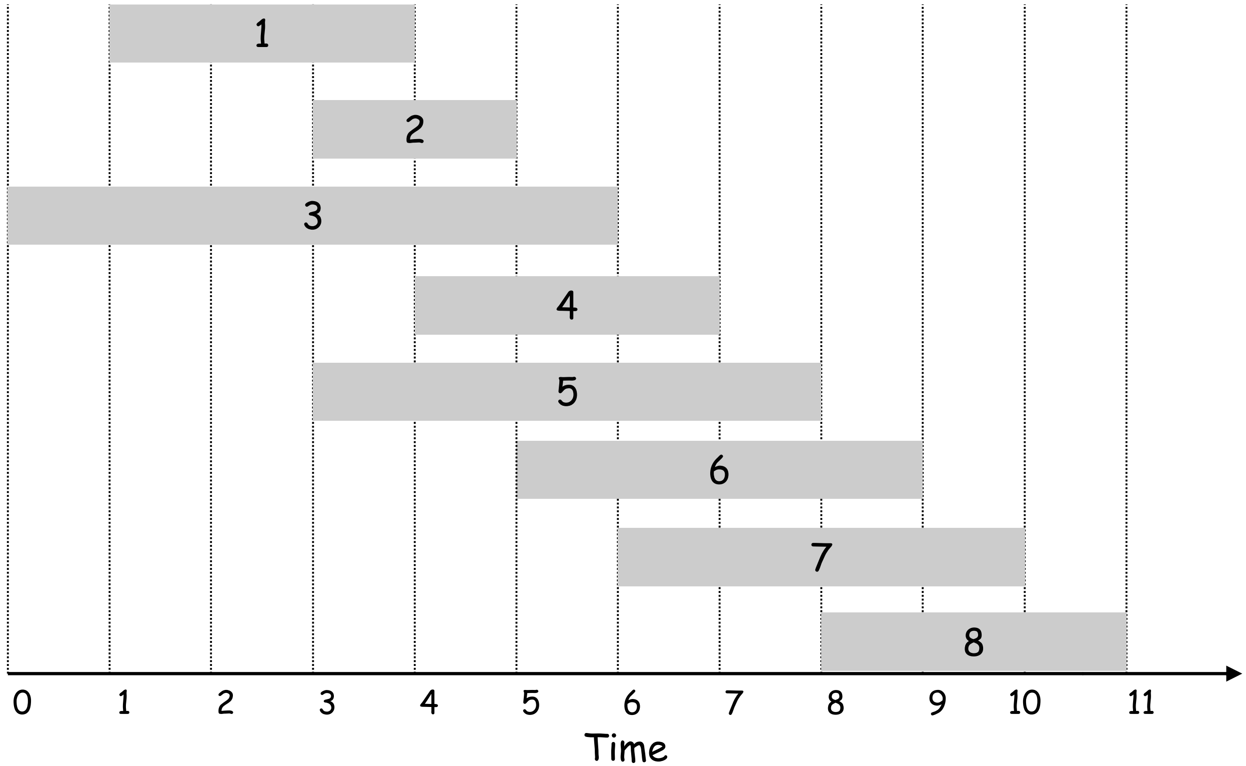

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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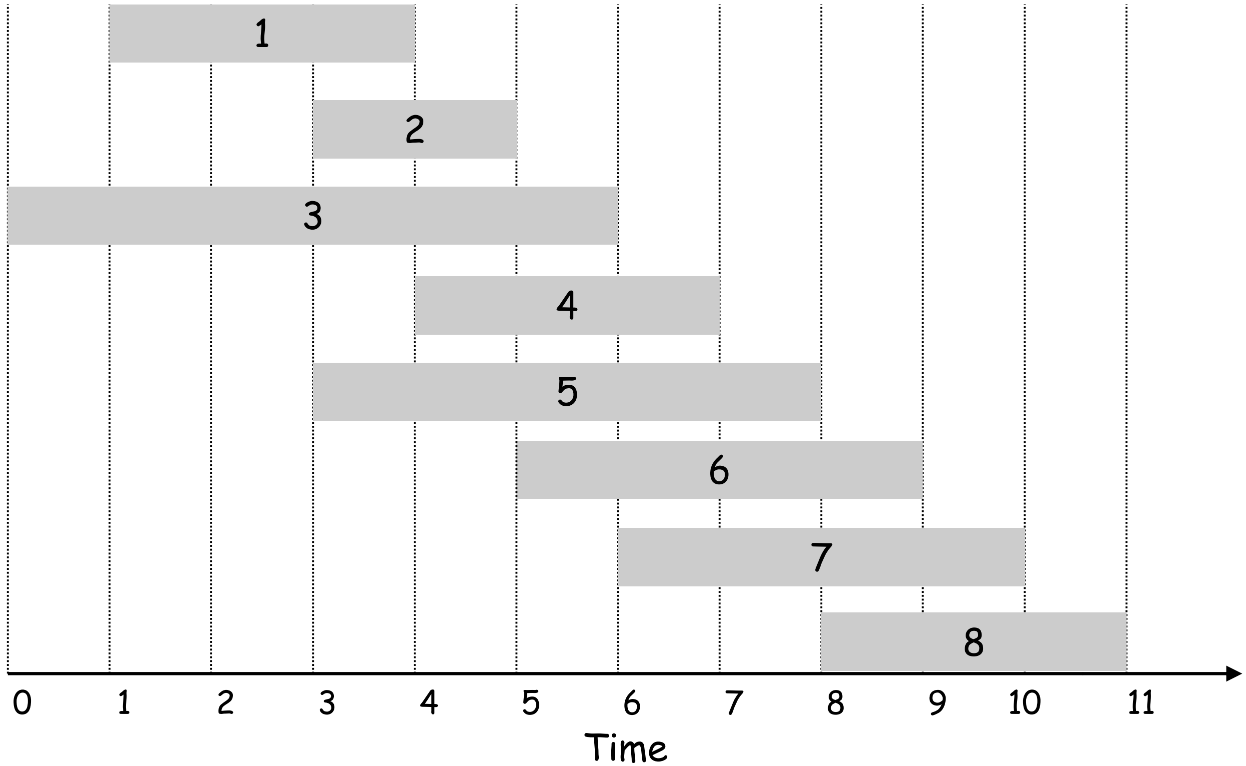

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	**4**
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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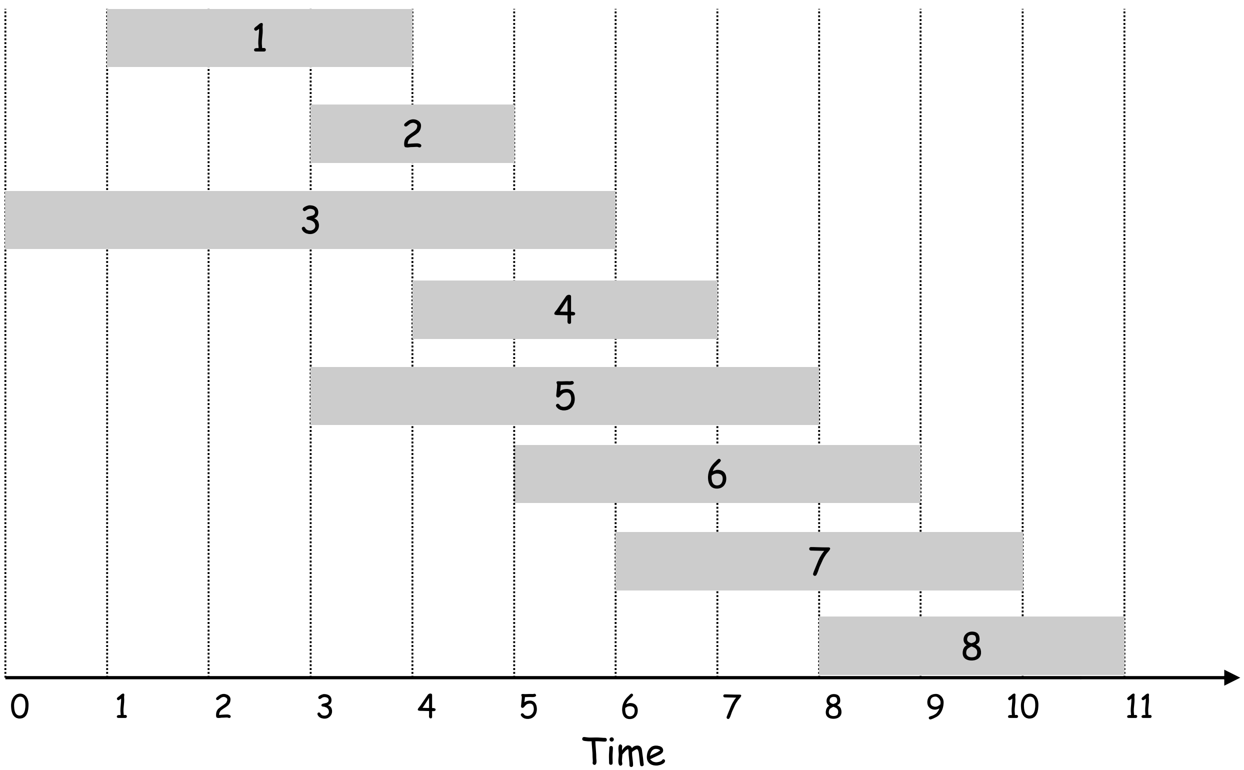

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	**4**
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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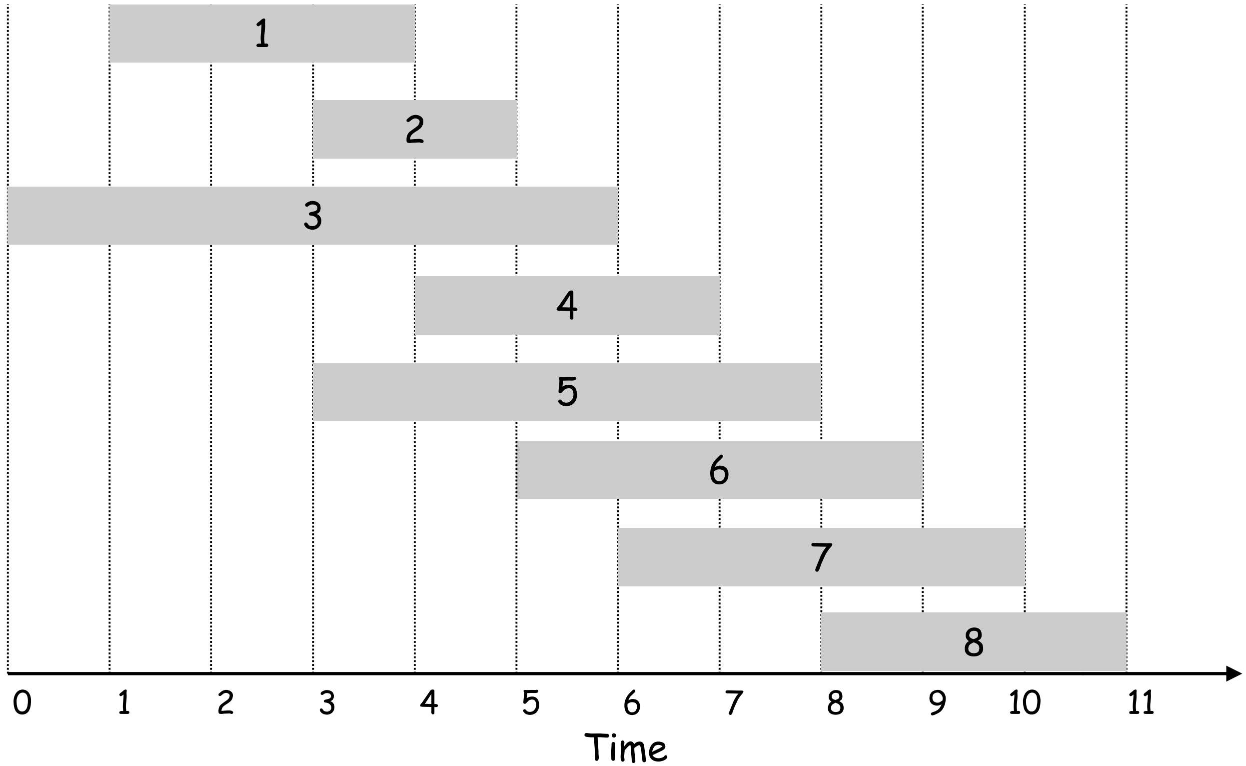


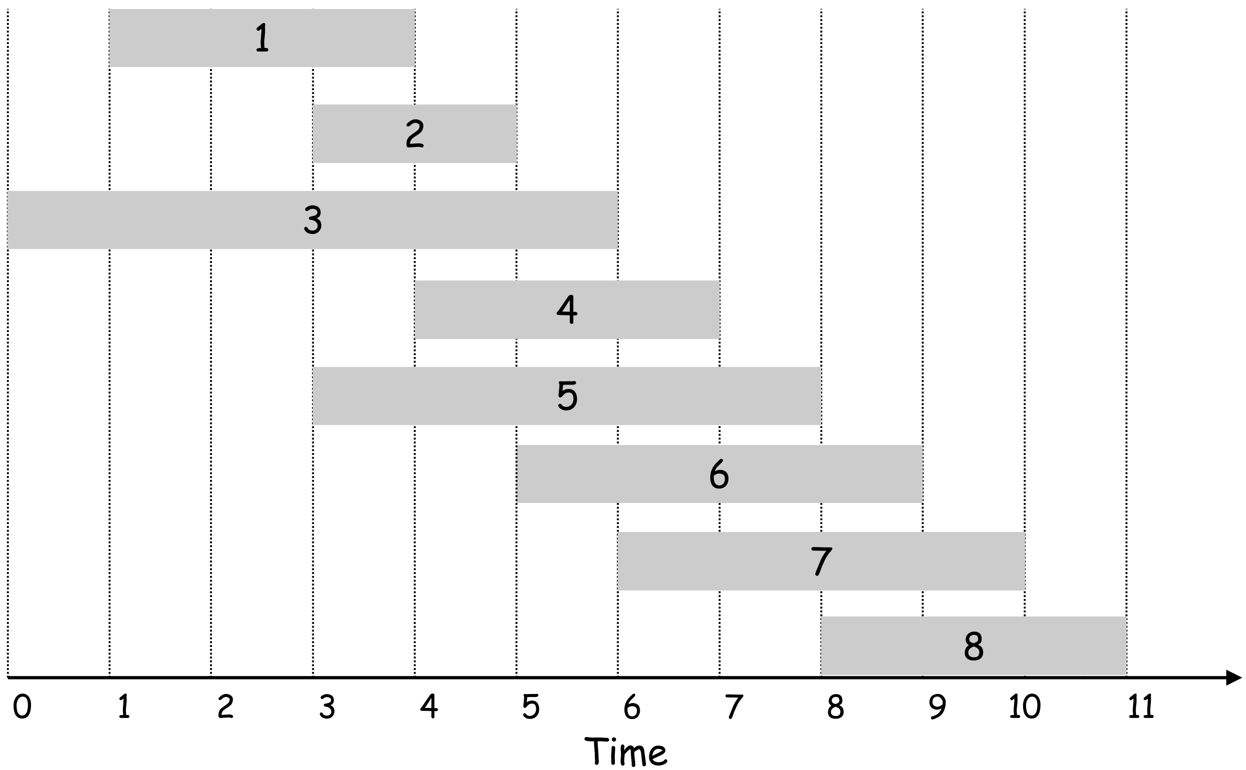
j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	**6**
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

$p(j) =$ largest index $i < j$ such that job i is compatible with j.



j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	**6**
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

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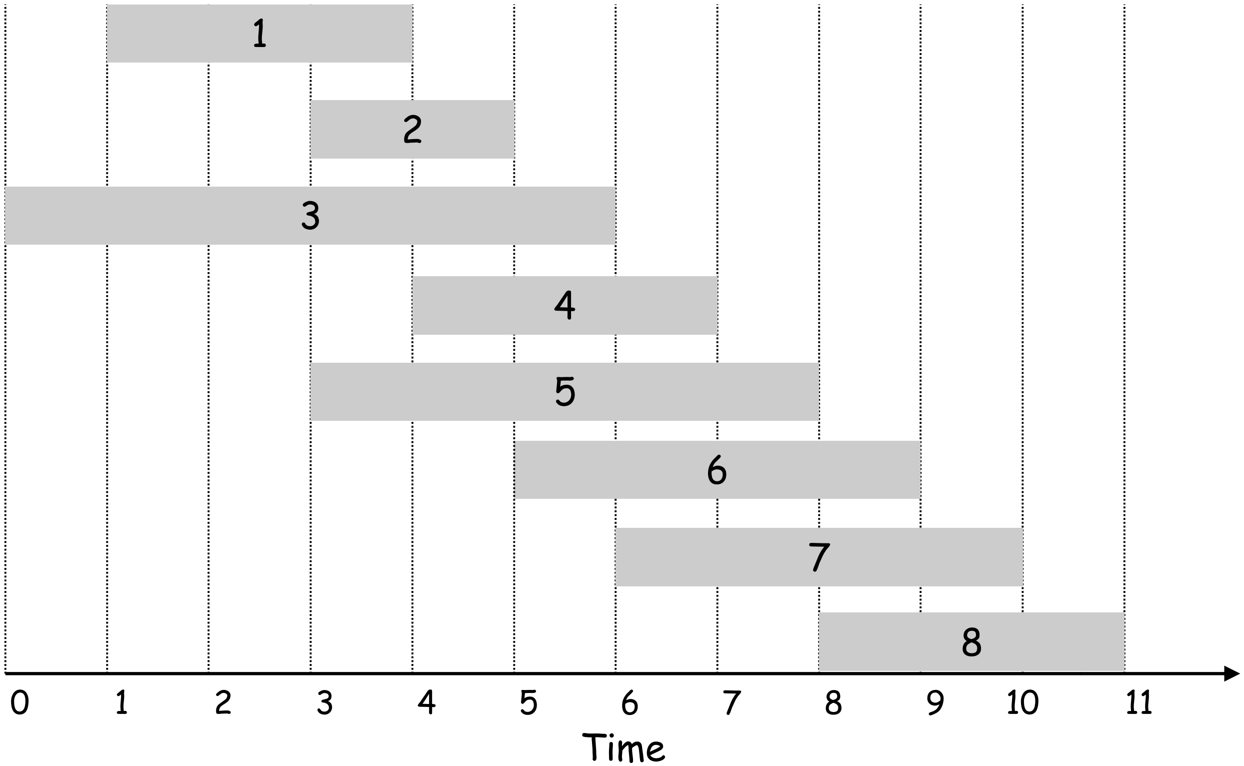


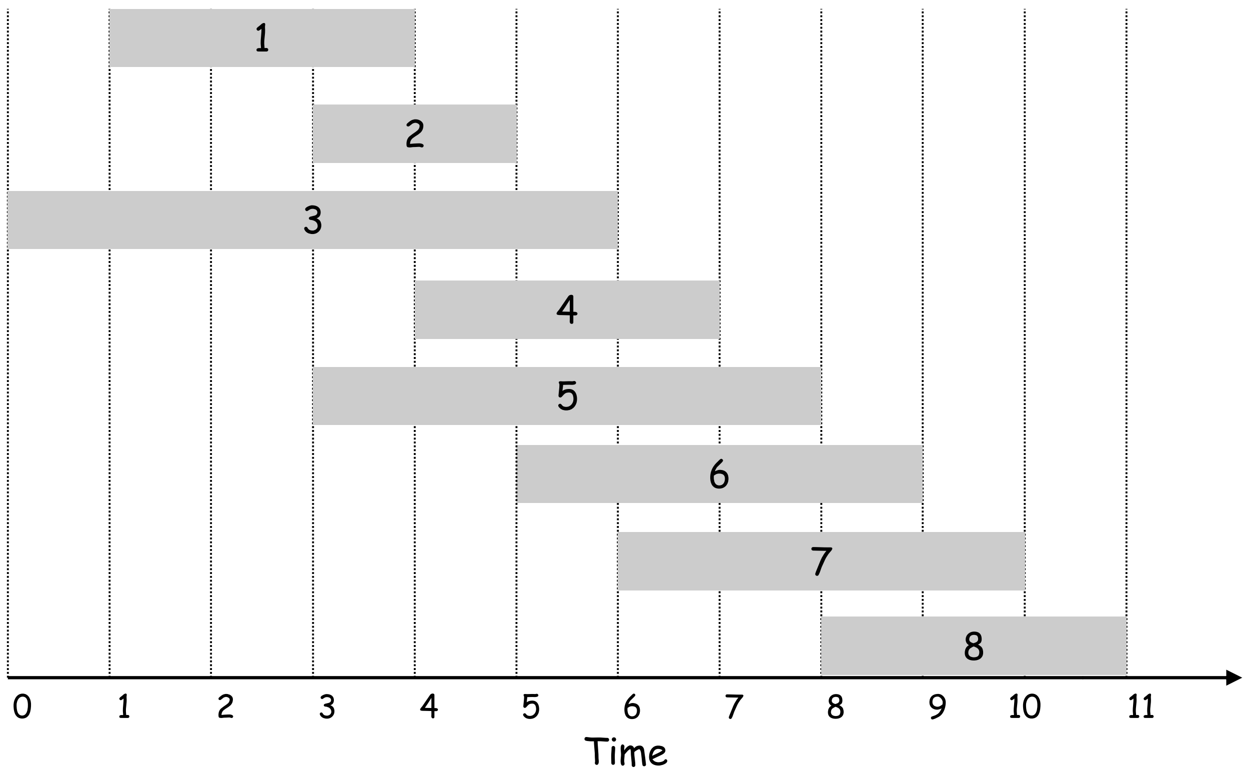
j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

$p(j) =$ largest index $i < j$ such that job i is compatible with j.



j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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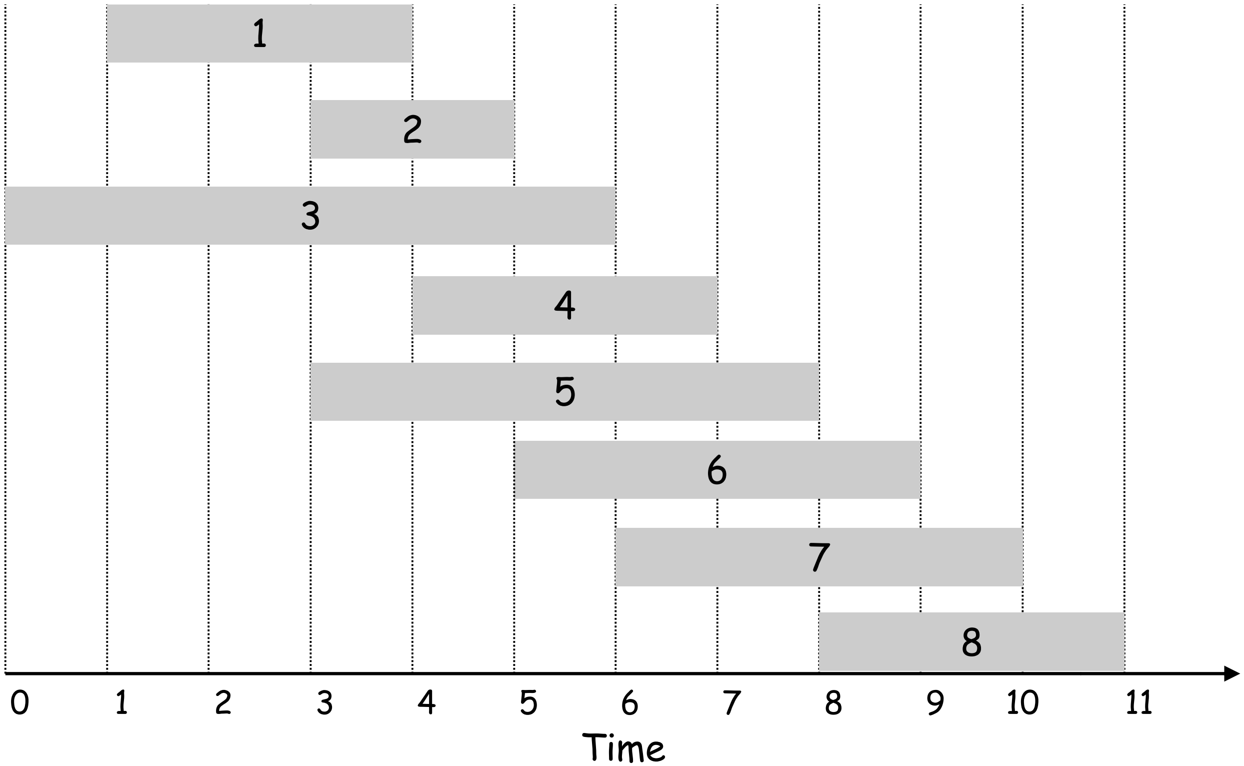

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	**10**

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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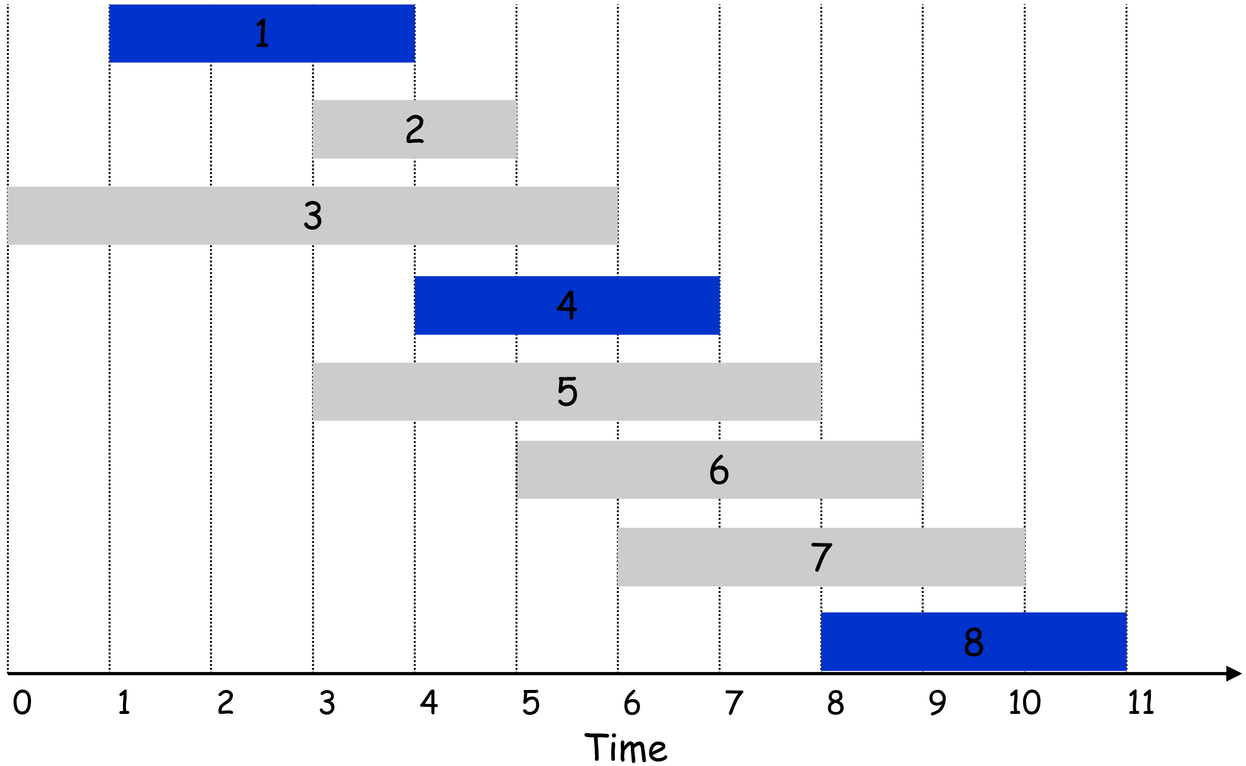

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	**10**

Dynamic Programming

- Give a solution of a problem using smaller (overlapping) sub-problems where

  the parameters of all sub-problems are determined in-advance

- Useful when the same subproblems show up again and again in the solution.

Knapsack Problem

Knapsack Problem

Given n objects and a "knapsack."

Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.

Knapsack has capacity of W kilograms.

Goal: fill knapsack so as to maximize total value.

Ex: OPT is { 3, 4 } with value 40.

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i.

Ex: { 5, 2, 1 } achieves only value = 35 $\Rightarrow$ greedy not optimal.

Dynamic Programming: First Attempt

Let $OPT(i)$ = Max value of subsets of items $1, \ldots, i$ of weight $\leq W$.

Case 1: $OPT(i)$ does not select item i
- In this case $OPT(i) = OPT(i-1)$

Case 2: $OPT(i)$ selects item i
- In this case, item i does not immediately imply we have to reject other items
- The problem does not reduce to $OPT(i-1)$ because we now want to pack as much value into box of weight $\leq W - w_i$

Conclusion: We need more subproblems, we need to strengthen IH.

Stronger DP (Strengthening Hypothesis)

Let $OPT(i, w)$ = Max value of subsets of items $1, \ldots, i$ of weight $\leq w$

Case 1: $OPT(i, w)$ selects item i

- In this case, $OPT(i, w) = v_i + OPT(i-1, w-w_i)$

Case 2: $OPT(i, w)$ does not select item i

- In this case, $OPT(i, w) = OPT(i-1, w)$.

Take best of the two

Therefore,

$$OPT(i, w) = \begin{cases} 0 & \text{If } i = 0 \\ OPT(i-1, w) & \text{If } w_i > w \\ \max(OPT(i-1, w), v_i + OPT(i-1, w-w_i)) & \text{o.w.,} \end{cases}$$

DP for Knapsack

```
Compute-OPT(i,w)
   if M[i,w] == empty
     if (i==0)
       M[i,w]=0
     else if (wi > w)
       M[i,w]=Comp-OPT(i-1,w)
     else
       M[i,w]= max {Comp-OPT(i-1,w), vi + Comp-OPT(i-1,w-wi)}
   return M[i, w]
```

recursive

```
for w = 0 to W
   M[0, w] = 0
for i = 1 to n
   for w = 1 to W
      if (wi > w)
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max {M[i-1, w], vi + M[i-1, w-wi ]}

return M[n, W]
```

Non-recursive

DP for Knapsack

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
φ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0											
{ 1, 2 }	0											
{ 1, 2, 3 }	0											
{ 1, 2, 3, 4 }	0											
{ 1, 2, 3, 4, 5 }	0											

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

```
if (w_i > w)
  M[i, w] = M[i-1, w]
else
  M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}
```

DP for Knapsack

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0											
{ 1, 2, 3 }	0											
{ 1, 2, 3, 4 }	0											
{ 1, 2, 3, 4, 5 }	0											

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

```
if (w_i > w)
  M[i, w] = M[i-1, w]
else
  M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}
```

DP for Knapsack

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
φ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7								
{ 1, 2, 3 }	0	1										
{ 1, 2, 3, 4 }	0	1										
{ 1, 2, 3, 4, 5 }	0	1										

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

```
if (w_i > w)
  M[i, w] = M[i-1, w]
else
  M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}
```

DP for Knapsack

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
φ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19					
{ 1, 2, 3, 4 }	0	1										
{ 1, 2, 3, 4, 5 }	0	1										

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
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```
if (w_i > w)
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```

DP for Knapsack

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29		
{ 1, 2, 3, 4, 5 }	0	1										

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

```
if (w_i > w)
  M[i, w] = M[i-1, w]
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  M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}
```

DP for Knapsack

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

```
if (w_i > w)
  M[i, w] = M[i-1, w]
else
  M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}
```

Knapsack Problem: Running Time

Running time: $\Theta(n \cdot W)$

- Not polynomial in input size!
- "Pseudo-polynomial.“
- Decision version of Knapsack is NP-complete.

Knapsack approximation algorithm:

There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% of optimum in time Poly(n,$\log W$).

UW Expert in similar problems

DP Ideas so far

- You may have to define an ordering to decrease #subproblems
- You may have to strengthen DP, equivalently the induction, i.e., you have may have to carry more information to find the Optimum.
- This means that sometimes we may have to use two dimensional or three dimensional induction