CSE 421

Divide and Conquer / Median

Yin Tat Lee

Exercise for Midterm

Given a weighted directed acyclic graph with n vertices and m edges. Give an $O(n+m)$ time algorithm to find the shortest path distance from vertex s to all other vertices. (Hints: Topological sort.)

Exercise for Midterm

Given a connected graph with n vertices and m edges with $m \geq n$. Give an $O(n)$ time algorithm to find a cycle.

Exercise for Midterm

Given an array of elements $a_1, a_2, \cdots, a_n$. Suppose that there are more than half of the elements are same. We call that element as the majority. Given an $O(n \log n)$ time algorithm to find the majority.
(Hints: Divide and Conquer)

Median

Selecting k-th smallest

Problem: Given numbers $x_1, \dots, x_n$ and an integer $1 \leq k \leq n$ output the k-th smallest number

$$\mathrm{Sel}(\{x_1, \dots, x_n\}, k)$$

A simple algorithm: Sort the numbers in time $O(n \log n)$ then return the k-th smallest in the array.

Can we do better?

Yes, in time $O(n)$ if $k = 1$ or $k = 2$.

Can we do $O(n)$ for all possible values of k?

An Idea

Choose a number w from $x_1, \dots, x_n$

Define

- $S_<(w) = \{x_i : x_i < w\}$
- $S_=(w) = \{x_i : x_i = w\}$
- $S_>(w) = \{x_i : x_i > w\}$

Can be computed in linear time

Solve the problem recursively as follows:

- If $k \le |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \le |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - |S_<(w)| - |S_=(w)|)$

Ideally want $|S_<(w)|, |S_>(w)| \le n/2$. In this case ALG runs in $O(n) + O\left(\frac{n}{2}\right) + O\left(\frac{n}{4}\right) + \cdots + O(1) = O(n)$.

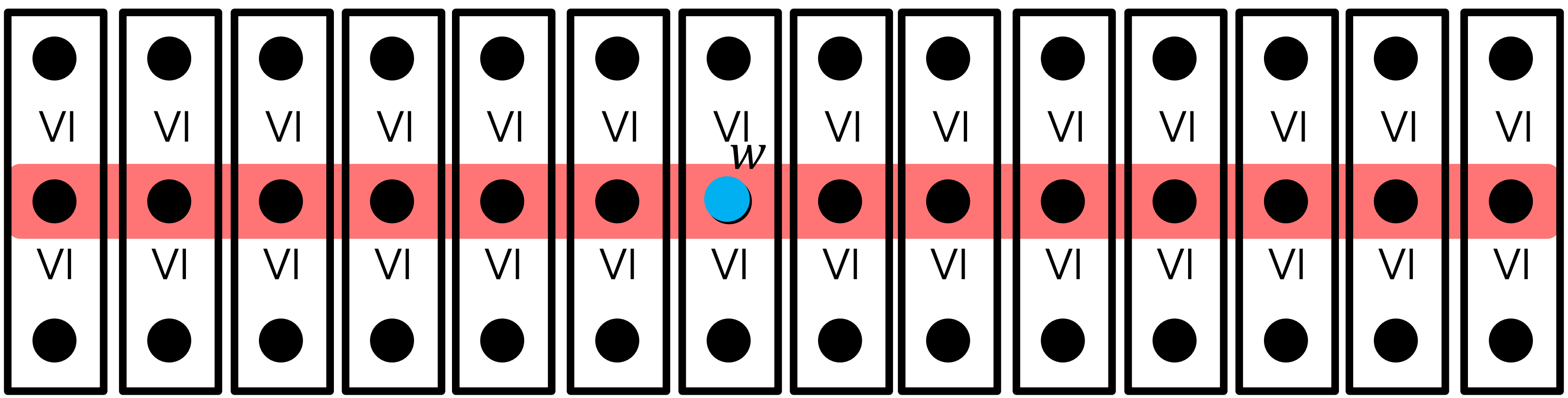
How to choose w?

Suppose we choose w uniformly at random
similar to the pivot in quicksort.

Then, $\mathbb{E}[|S_<(w)|] = \mathbb{E}[|S_>(w)|] = n/2$. Algorithm runs in $O(n)$ in expectation.

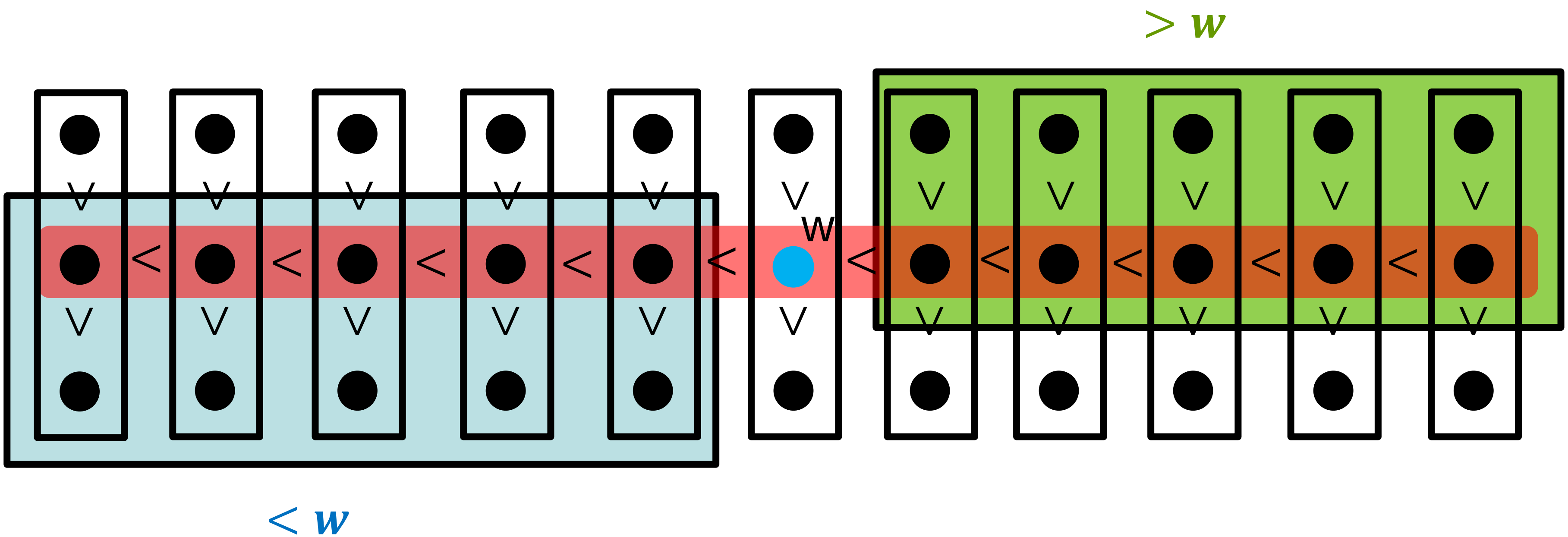
Can we get $O(n)$ running time deterministically?

- Partition numbers into sets of size 3.
- Sort each set (takes O(n))
- $w = Sel(midpoints, n/6)$



Assume all numbers are distinct for simplicity.

How to lower bound $|S_<(w)|, |S_>(w)|$?



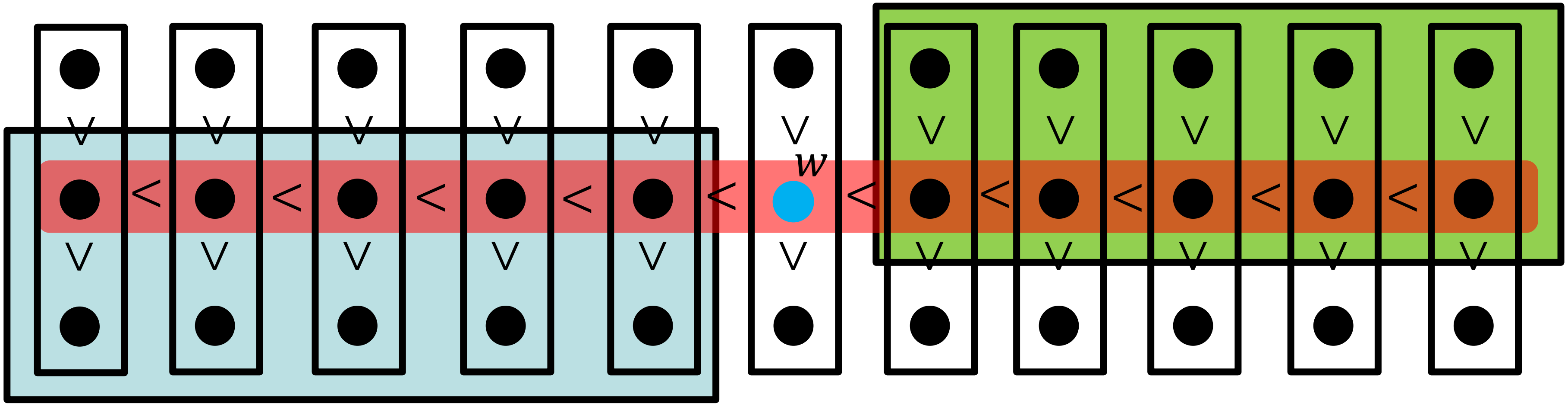
- $|S_<(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$
- $|S_>(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$.

$$\frac{n}{3} \leq |S_<(w)|, |S_>(w)| \leq \frac{2n}{3}$$

So, what is the running time?

Assume all numbers are distinct for simplicity.

Asymptotic Running Time?



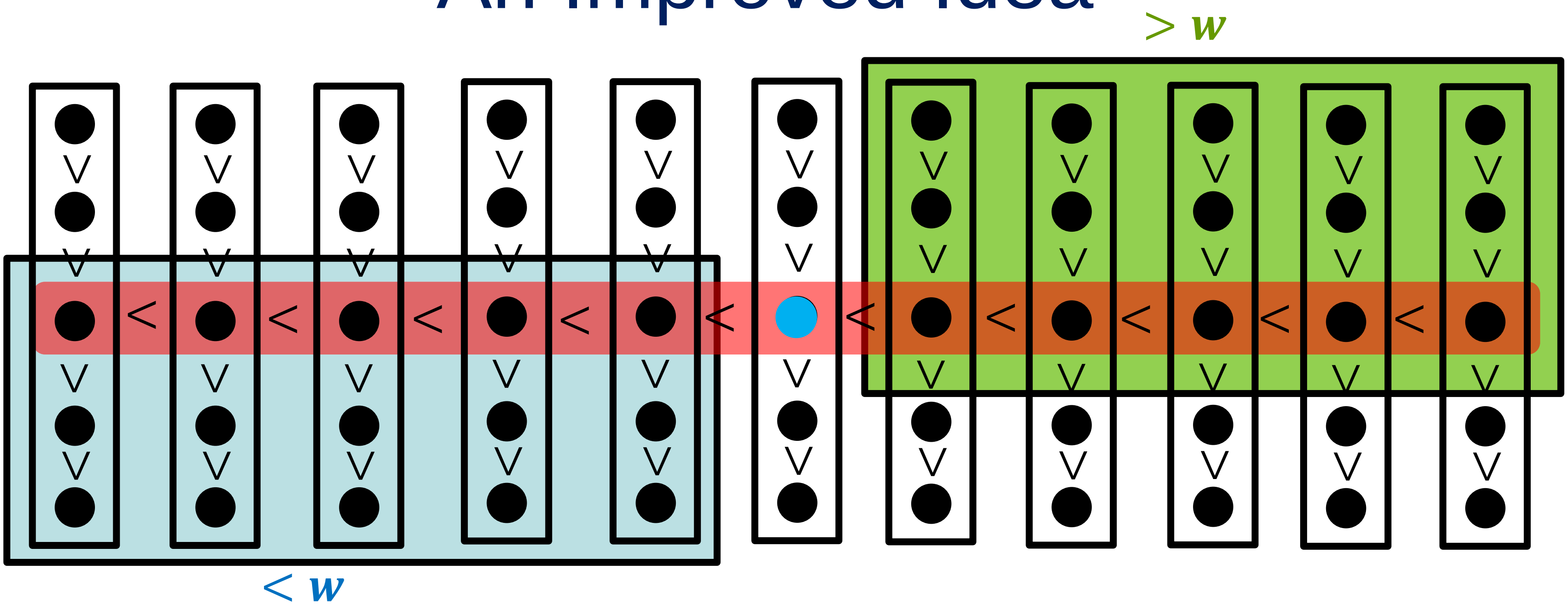
- If $k \leq |S_{<}(w)|$, output $Sel(S_{<}(w), k)$
- Else if $k \leq |S_{<}(w)| + |S_{=}(w)|$, output w
- Else output $Sel(S_{>}(w), k - |S_{<}(w)| - |S_{=}(w)|)$

$O(n \log n)$ again?
So, what is the point?

Where $\frac{n}{3} \leq |S_{<}(w)|, |S_{>}(w)| \leq \frac{2n}{3}$

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + O(n) \Rightarrow T(n) = O(n \log n)$$

An Improved Idea



Partition into n/5 sets. Sort each set and set $w = Sel(midpoints, n/10)$

- $|S_<(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$
- $|S_>(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$

$\Rightarrow$ $\frac{3n}{10} \leq |S_<(w)|, |S_>(w)| \leq \frac{7n}{10}$

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n) \Rightarrow T(n) = O(n)$$

An Improved Idea

Sel(S, k) {
    $n \leftarrow |S|$
    If (n < ??) return ??
    Partition S into n/5 sets of size 5
    Sort each set of size 5 and let M be the set of medians, so |M|=n/5
    Let w=Sel(M,n/10)
    For i=1 to n{
      If $x_i < w$ add x to $S_<(w)$
      If $x_i > w$ add x to $S_>(w)$
      If $x_i = w$ add x to $S_=(w)$
    }
    If $(k \leq |S_<(w)|)$
      return Sel$(S_<(w), k)$
    else if $(k \leq |S_<(w)| + |S_=(w)|)$
      return w;
    else
      return Sel$(S_>(w), k - |S_<(w)| - |S_=(w)|)$
}

We can maintain each set in an array